SOME PROBLEMS

(1) Let:

$$\alpha = \int_0^{\log(2)} x \log(e^x - 1) dx.$$

'Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. It is known that $[\mathbb{Q}(\alpha) : \mathbb{Q}] > 1$ but it is not even known whether or not $[\mathbb{Q}(\alpha) : \mathbb{Q}] = \infty$.

(2) There exists a finite group G that is not the Galois group of any Galois extension over $\mathbb{Q}$. The answer is known to be false (and a good exercise!) if G is abelian, and it is also known to be false if $G = S_n$ or $G = A_n$ or if G is solvable.

(3) For $d > 0$ an integer and:

$$\omega = \frac{1 + \sqrt{d}}{2}$$

define $\mathbb{Z}[d]$ to be the ring:

$$\mathbb{Z}[d] = \{a + \omega b : a, b \in \mathbb{Z}\}.$$

For which d is $\mathbb{Z}[d]$ a UFD? The answer is known for $d \leq 100$ or so. Note that $\mathbb{Z}[d]$ is a UFD if and only if it is a PID (not obvious!).

(4) Let $\mathbb{F}$ be a field of characteristic zero, $N > 0$ an integer, and $f_1, ..., f_N \in \mathbb{F}[x_1, ..., x_N]$. If:

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1} & \cdots & \frac{\partial f_N}{\partial x_N} \end{pmatrix}$$

is a nonzero constant then $(x_1, ..., x_N) \mapsto (f_1(x_1, ..., x_N), ..., f_N(x_1, ..., x_N))$ is invertible with inverse equal to:

$$(x_1, ..., x_N) \mapsto (g_1(x_1, ..., x_N), ..., g_N(x_1, ..., x_N))$$

for some $g_1, ..., g_N \in \mathbb{F}[x_1, ..., x_N]$.

(5) Let R be a ring or rng such that every left ideal and every right ideal is finitely generated and define:

$$J = \{y \in R : 1 + xy \text{ is invertible for all } x \in R\}.$$

Then:

$$\bigcap_{n \in \mathbb{N}} J^n = \{0\}.$$