MAT347 TUTORIAL 5

If G acts on a set X, then for any $x \in X$ recall that we define the **orbit** of x:

$$\mathcal{O}_x = \{g \cdot x \mid g \in G\}$$

and the **stabilizer** of x:

$$\mathrm{Stab}_x = \{g \in G : g \cdot x = x\}.$$

(1) Suppose G acts on a set X and let $Y \subseteq X$. Give a necessary and sufficient condition so that the action of G on X restricts to an action of G on Y.
(2) Using the previous exercise prove that if G acts on a set X then the action always restricts to an action on its orbits.
(3) Prove that conjugation $(g, h) \mapsto ghg^{-1}$ defines a left group action of any group G on itself.
(4) Prove that conjugation $(g, h) \mapsto g^{-1}hg$ defines a right group action of any group G on itself.
(5) What are the orbits of the two previously defined actions? What are the stabilizers?
(6) For which groups is this action transitive? Free? Faithful? Trivial?
(7) Let $G = \mathrm{GL}_2(\mathbb{C})$ be the group of invertible 2×2 complex matrices. Describe all the orbits and stabilizers of the conjugation action.
(8) Let $G = \mathrm{SU}(2)$ be the group of 2×2 unitary matrices with determinant 1. Describe all the orbits and stabilizers of the conjugation action.
(9) Let $G = \mathrm{SL}_2(\mathbb{R})$ be the group of 2×2 matrices with determinant 1. Describe all the orbits and stabilizers of the conjugation action.
(10) Let V be a finite dimensional vector space and $\mathrm{GL}(V)$ be the group of invertible linear transformations $V \to V$. Prove that the map $(T, v) \mapsto T(v)$ is a left group action (called the **left regular action**).
(11) What are the orbits and stabilizers of the left regular action?