MAT347 TUTORIAL 4

(1) Prove or disprove: the center of $G/\mathrm{Cent}(G)$ is trivial.
(2) Fix a prime p and define $G_p = \{z \in \mathbb{C} : z^{p^n} = 1 \text{ for some n} \in \mathbb{Z}\}$. Prove that G_p is an infinite group such that for all $H \leq G_p$ we have $H = G_p$ or else $H \cong \mathbb{Z}/k\mathbb{Z}$ for some k.
(3) With G_p as in the previous exercise prove that $\phi \colon G_p \to G_p$, $\phi(z) = z^p$ is a group homomorphism and moreover that $G_p \cong G_p/\ker(\phi)$.
(4) Give an example of a group G and a normal subgroup N with more than one element such that the quotient map $G \to G/N$ is a group isomorphism.

Let $G = \mathrm{SL}_2(\mathbb{R})$ be the group of invertible 2×2 real matrices with determinant 1 under composition.

(5) Prove that for all $g \in G$ there exists an orthogonal matrix k and an upper triangular u such that $g = ku$. (Hint: start with polar decomposition)
(6) Prove that for all $g \in G$ there exists an orthogonal matrix k a diagonal matrix a and a unipotent matrix n such that $g = kan$ (where *unipotent* means upper triangular with 1s on the diagonal).
(7) Prove that for all $g \in G$ there exists $h \in G$ such that hgh^{-1} is either orthogonal, diagonal, or unipotent.
(8) Prove that $\mathrm{Cent(G)} = \{\pm I\}$.
(9) Prove that the center of $G/\mathrm{Cent(G)}$ is trivial.