MAT347 TUTORIAL

- Let R be a PID and let T_1, T_2 be torsion R-modules. Suppose $R^k \oplus T_1 \cong R^m \oplus T_2$. Prove that $k = m$ and $T_1 \cong T_2$. (Hint: you have seen this before)
- Let M, N be R-modules and $\varphi\colon M \to N$ an R-module homomorphism. Prove that $\mathrm{Ann}(v) \subseteq \mathrm{Ann}(\varphi(v))$ for all $v \in M$. Deduce that $\mathrm{Ann}(v) = \mathrm{Ann}(\varphi(v))$ when φ is an isomorphism.
- Let R be a PID and $x \in R$ a nonzero non unit. Let $v \in R/(x)$ be a generator as an R-module. What is $\mathrm{Ann}(v)$?
- Let R be a PID and $x, y \in R$ be nonzero non units. Suppose $\varphi\colon R/(x) \to R/(y)$ is an isomorphism. Use the previous two problems to show that $(x) = (y)$. Finally, conclude that $x \sim y$.

Let V be a finite dimensional vector space over a field $\mathbb{F}$. Let $\mathrm{End}(V)$ denote linear transformations $V \to V$ and let $GL(V) \subset \mathrm{End}(V)$ denote the invertible ones. Also let $\mathrm{Mat}_n(\mathbb{F})$ denote $n \times n$ matrices and let $GL_n(\mathbb{F}) \subset \mathrm{Mat}_n(\mathbb{F})$ denote the invertible ones.

- Prove that $\mathrm{Mat}_n(\mathbb{F})$ is a vector space over $\mathbb{F}$ of dimension n^2. Prove that $\mathrm{End}(V)$ is a vector space over $\mathbb{F}$ of dimension $\dim(V)^2$. If $\dim(V) = n$ prove that an isomorphism $\mathrm{End}(V) \to \mathrm{Mat}_n(\mathbb{F})$ is equivalent to the choice of basis for V. Show also that a basis for V naturally defines a basis for $\mathrm{End}(V)$.
- For $g \in GL(V)$ prove the map $\mathrm{Ad}_g\colon \mathrm{End}(V) \to \mathrm{End}(V)$ given by $\mathrm{Ad}_g(T) = g \circ T \circ g^{-1}$ is linear.
- For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F})$ consider the map $\mathrm{Ad}_g\colon \mathrm{Mat}_2(\mathbb{F}) \to \mathrm{Mat}_2(\mathbb{F})$ given by $\mathrm{Ad}_g(M) = gMg^{-1}$. Let β denote the standard basis for $\mathrm{Mat}_2(\mathbb{F})$. Find the matrix $[\mathrm{Ad}_g]_\beta^\beta$ (it is a 4×4 matrix).
- Let V be dimension n, let $g \in GL(V)$, let β_0 be a basis for V and β the induced basis for $\mathrm{End}(V)$. find the matrix $[\mathrm{Ad}_g]_\beta^\beta$ (it is an $n^2 \times n^2$ matrix).
- Suppose V_1, V_2 are vector spaces over $\mathbb{F}$ and let $T_i \in \mathrm{End}(V_i)$. Show there exists a unique linear transformation called their *tensor product* denoted by $T_1 \otimes T_2 \in \mathrm{End}(V_1 \otimes V_2)$ such that on simple tensors we have $T_1 \otimes T_2(v_1 \otimes v_2) = T_1(v_1) \otimes T_2(v_2)$.
- If $\dim(V_1), \dim(V_2) > 1$ prove that there exist linear transformations $T \in \mathrm{End}(V_1 \otimes V_2)$ that are not of the form of the previous question (i.e. not equal to $T_1 \otimes T_2$).
- Recall from homework that $\mathrm{End}(V) \cong V \otimes V^*$. For $g \in GL(V)$ we can then view Ad_g as a map $\mathrm{Ad}_g \in \mathrm{End}(V \otimes V^*)$. When viewed this way, prove that Ad_g is a tensor product of two linear maps $V \to V$ and $V^* \to V^*$. (Hint: the first one is g).
- Compare the tensor product representation of Ad_g with the matrix representation worked out previously.