MAT347 TUTORIAL

- True or false: $I \subset R$ is a prime ideal if and only if $R \setminus I$ is multiplicative.
- Let R be a commutative domain and $S \subset R \setminus 0$ multiplicative. State a universal property for the localization $S^{-1}R$ of S at R similar to the one given in class for the field of fractions $Q(R)$ (there exists an inclusion map $\iota \colon R \to Q(R)$ such that for any field $\mathbb{F}$ and morphism $\alpha \colon R \to \mathbb{F}$ there is a unique $\widehat{\alpha} \colon Q(R) \to \mathbb{F}$ such that $\alpha = \widehat{\alpha} \circ \iota$). (Hint: you don't need to assume anything is a field).
- Taking the universal property as the definition of localization, decide whether or not we generalize to allow subsets containing 0. What about allowing zero divisors in R and S?
- Prove that a ring is an integral domain if and only if it is a subring of a field. (Hint: this should be easy).
- If R is a commutative domain and $S \subset R \setminus 0$ is multiplicative use the previous problem to prove that $S^{-1}R$ is a commutative domain.
- Suppose R is commutative, fix $r \in R \setminus 0$, and define $S_r = \{r^n : n \geq 0\}$. Use the universal property to prove $S^{-1}R \cong R[x]/(rx - 1)$.
- Let R be a commutative ring. Show the following are equivalent:
  (1) R has a unique maximal ideal.
  (2) If a, b are not units then $a + b$ is not a unit.
  (3) For any $r \in R$ either r or $1 - r$ is a unit.
  A commutative ring satisfying the above (equivalent) properties is called **local**.
- For R a commutative ring and $I \subset R$ a prime ideal let $S = R \setminus I$. Prove that $S^{-1}R$ is local.
- Give an example of a finite as well as an infinite ring neither of which is local.
- Classify all cyclic local rings.
- Let X be a topological space and fix $p \in X$. For $f, g \colon X \to \mathbb{R}$ define $f \sim g$ if there exists an open neighbourhood U of p such that $f|_U = g|_U$. Prove that $C(X)/\sim$ is a local ring. What is the multiplicative identity? What are the units?