MAT347 TUTORIAL

- Let $\mathbb{F}$ be a field and for $n \geq 1$ an integer define the $\mathbb{F}[x]$-module M_n by $M_n = \mathbb{F}^n$ as an abelian group and $x \in \mathbb{F}[x]$ acts on $\mathbb{F}^n$ by left multiplication by the following $n \times n$ matrix:
$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$
(i.e. a single Jordan block with generalized eigenvalue 0). Compute $M_n \otimes_{\mathbb{F}[x]} M_m$.
- Prove that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} M = 0$ for any torsion $\mathbb{Z}$-module M
- Prove that $R \otimes_R M \cong M$ for any R-module M.
- True/False: a subring of a PID is a PID.
- Give an example of a simple $\mathbb{Q}[x]$-module that has dimension 2 as a $\mathbb{Q}$-module where simple means the only submodules of M are 0 and M.
- For M an R-module and $I < R$ an ideal define $M[I] = \{m \in M : rm = 0 \ \forall r \in R\}$. Prove that $M[I]$ is an R-module and that $\mathrm{Hom}_R(R/I, M) \cong M[I]$ as R-modules.
- Define the ring $R = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 : a_i \in \mathbb{Z}\}$.
  1. Find $R^{\times}$.
  2. Show that if $p \in \mathbb{Z}$ is prime then $p \in R$ is prime.
  3. Show that for $f \in R$ irreducible in $\mathbb{Q}[x]$ and $f(0) = \pm 1$ then f is prime in R.
  4. Prove that R is not a UFD.