MAT347 TUTORIAL

(1) Prove or disprove: If R is an integral domain with ideals $I, J \trianglelefteq R$, if $R/I \cong R/J$ as R-modules then $I = J$.

(2) Let R be a commutative ring and let M be an R-module. Show that $M \cong R/I$ as R-modules for some maximal ideal $I \trianglelefteq R$ if and only if the only submodules of M are 0 and M (such modules are called *simple*).

(3) Prove that there exists an integral domain R and a torsion R-module M such that $\mathrm{Ann}_R(M) = 0$ where:

$$\mathrm{Ann}_R(M) = \{r \in R : rm = 0 \text{ for all } m \in M\}$$

and (c.f. homework 11) a module is called **torsion** if for all $m \in M$ there exists $r \in R$ such that $rm = 0$.

(Hint: first decide if M can be finitely generated).

(4) Let M be an abelian group such that $M \cong \mathbb{Z}^n$ for some $n \geq 1$ (as abelian groups) and suppose there exists a group homomorphism $f \colon M \to M$ such that $f \circ f = -\mathrm{Id}$.

(a) Show that M can be given the structure of a $\mathbb{Z}[i]$ module by the formula $(a + bi) \cdot m = am + bf(m)$.

(b) Show that:

$$M \cong \bigoplus_{i=1}^{k} \mathbb{Z}[i]$$

for some k and deduce that n is even.

(c) Prove that there exists a basis $\{e_1, ..., e_n\}$ for M as a $\mathbb{Z}$-module such that for all $1 \leq j \leq n$ we have $f(e_{2j-1}) = e_{2j}$ and $f(e_{2j}) = -e_{2j-1}$.

(d) If $g \colon M \to M$ is another group homomorphism satisfying $g \circ g = -\mathrm{Id}$ prove that there exists a group isomorphism $\phi \colon M \to M$ such that $\phi \circ f = g \circ \phi$.

(5) True or false: there exists a ring R such that $R \cong R \oplus R$ as left R-modules. (Hint: can R be commutative?)