Homework Assignment 16

Solve and submit your solutions of the following problems. Note also that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Due date

Friday, March 20, 2026 11:59 pm (Eastern Daylight Time)

Late penalty

5% deducted per hour

Q1 (20 points)

1. By elementary means, prove that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$
2. Show that $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)/\mathbb{Q}$ is a splitting extension of degree 4.
3. Identify $G = \operatorname{Gal}\big(\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)/\mathbb{Q}\big)$.
4. Draw the lattice of subfields of $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$ and the corresponding lattice of subgroups of its Galois group G.

Q2 (30 points)

Let $\omega := \exp(2\pi i/7)$ be a 7th root of 1.

1. Possibly using Q5 of HW15, show that $[\mathbb{Q}(\omega) : \mathbb{Q}] = 6$.
2. Show that $\mathbb{Q}(\omega)/\mathbb{Q}$ is a splitting extension.
3. Using the automorphism ϕ of $\mathbb{Q}(\omega)/\mathbb{Q}$ defined by extending $\omega \mapsto \omega^3$, show that $G = \operatorname{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ is isomorphic to the cyclic group of order 6, C_6.

4. Draw the lattice of subgroups of the Galois group G.
5. Draw the corresponding lattice of subfields of $\mathbb{Q}(\omega)$.
6. I believe that $\alpha := \omega + \omega^6$ and $\beta := \omega + \omega^2 + \omega^4$ are primitive elements of the two interesting subfields of $\mathbb{Q}(\omega)$. Prove me right!
7. How did I come up with α and β?

Q3 (20 points)

Let $E := \mathbb{Q}(x^4 + 1)$.

1. Identify $G = \mathrm{Gal}(E/\mathbb{Q})$.
2. Find all the subfields of E.
3. Find the automorphisms of E whose fixed fields are $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{-2})$, and $\mathbb{Q}(i)$.
4. Is there an automorphism of E whose fixed field is $\mathbb{Q}$?

Q4 (10 points)

Let E/F be a finite extension of a field F whose characteristic is 0 (not necessarily a splitting extension!). Show that there are only finitely many fields in between F and E.

Ready to submit?

→ Please ensure all pages are in order and rotated correctly before you submit

→ You will not be able to resubmit your work after the due date has passed.

Please wait...