Homework Assignment 14

Solve and submit your solutions of the following problems. Note also that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Due date

Friday, February 27, 2026 11:59 pm (Eastern Standard Time)

Late penalty

5% deducted per hour

Q1 (10 points)

Suppose f and g are irreducible polynomials over a field F whose degrees are greater than 1 and relatively prime. If a is a zero of f in some extension of F, show that g does not have a root in $F(a)$.

Q2 (10 points)

Find the degree and a basis for $\mathbb{Q}\left(\sqrt{3}+\sqrt{5}\right)/\mathbb{Q}\left(\sqrt{15}\right)$ and for $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}\right)/\mathbb{Q}$.

Q3 (10 points)

Find an example of an extension E/F and elements $a, b \in E$ such that $F(a) \neq F(a, b) \neq F(b)$ and $[F(a, b) : F] < [F(a) : F][F(b) : F]$.

Q4 (10 points)

Find $\text{minpoly}_{\mathbb{Q}}\left(\sqrt[3]{2} + \sqrt[3]{4}\right)$.

Q5 (10 points)

Let $a \in E/F$. Show that $[F(a) : F(a^3)] \leq 3$ and show by examples that $[F(a) : F(a^3)]$ can be 1, 2, or 3.

Q6 (10 points)

Suppose that $[E : \mathbb{Q}] = 2$. Show that there is an integer d such that $E = \mathbb{Q}[\sqrt{d}]$ and d is not divisible by the square of any prime.

Q7 (0 points)

If you're in the mood, also solve the following questions, but do not submit your solutions. Note that some of these questions may be regarded as "warmups" for Q1-Q6:

1. Show that a field F is algebraically closed iff every irreducible polynomial in $F[x]$ is linear.
2. Let $E, m \in \mathbb{Q}$ with $m \neq 0$. Show that $\mathbb{Q}\left(\sqrt{E}\right) = \mathbb{Q}\left(\sqrt{m}\right)$ iff there is some $c \in \mathbb{Q}$ such that $E = mc^2$.
3. Suppose $[E : F]$ is prime. Show that for any $a \in E$, $F(a) = F$ or $F(a) = E$.
4. Let $a, b \in E/F$ and assume a and b are algebraic over F of degrees m and n respectively, where $\gcd(m, n) = 1$. Show that $[F(a, b) : F] = mn$.
5. Find $\text{minpoly}_{\mathbb{Q}}\left(\sqrt{-3} + \sqrt{2}\right)$.
6. Let $0 \neq f \in F[x]$ and let $a \in E/F$. Show that if $f(a)$ is algebraic over F then so is a.
7. Show that $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{2}\right) = \mathbb{Q}\left(\sqrt[6]{2}\right)$.