This is a preview of what students will see when they are submitting the assignment. Interactive features are disabled.

Homework Assignment 13

Solve and submit your solutions of the following problems. Note also that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Due date

Friday, February 13, 2026 11:59 pm (Eastern Standard Time)

Late penalty

5% deducted per hour

Q1 (10 points)

Let M be a module over a PID R. Assume that M is isomorphic to $R^k \oplus R/\langle a_1 \rangle \oplus R/\langle a_2 \rangle \oplus \cdots \oplus R/\langle a_l \rangle$, with a_i non-zero non-units and with $a_1 \mid a_2 \mid \cdots \mid a_l$. Assume also that M is isomorphic to $R^m \oplus R/\langle b_1 \rangle \oplus R/\langle b_2 \rangle \oplus \cdots \oplus R/\langle b_n \rangle$, with b_i non-zero non-units and with $b_1 \mid b_2 \mid \cdots \mid b_l$. Prove that $k = m$, that $l = n$, and that $a_i \sim b_i$ for each i.

Q2 (10 points)

Let q and p be primes in a PID R such that $p \nsim q$, let $\hat{p}$ denote the operation of "multiplication by p", acting on any R-module M, and let s and t be positive integers.

1. For each of the R-modules R, $R/\langle q^t \rangle$, and $R/\langle p^t \rangle$, determine $\ker \hat{p}^s$ and $(R/\langle p \rangle) \otimes_R \ker \hat{p}^s$.

2. Explain why this approach for proving the uniqueness in the structure theorem for finitely generated modules fails.

Q3 (10 points)

Show that if R is a PID and S is a multiplicative subset of R then $S^{-1}R$ is also a PID.

Q4 (10 points)

Write $x^3 + x^2 + x + 1$ as a product of primes in the ring $(\mathbb{Z}/2)[x]$.

Q5 (10 points)

1. Let $f = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ and assume that $a_n \neq 0$. Show that if $r/s \in \mathbb{Q}$ with r and s relatively prime satisfies $f(r/s) = 0$, then $r|a_0$ and $s|a_n$.
2. Formulate a version of this statement that is true over an arbitrary UFD R. No need to re-prove.

Q6 (0 points)

Just for fun (or maybe misery). Let x be some root of the equation

$$x^5 + (\sqrt[3]{2} - \sqrt{3})x^4 + \frac{1}{\sqrt{\sqrt{3} + \sqrt{5}}}x^3 - 1 = 0.$$

We know that x is algebraic over $\mathbb{Q}$. Find a polynomial f with rational coefficients whose roots include x. What is $\deg f$?

You will need to write some code to answer this question, or use some computer algebra system.

Why am I asking? This question demos that while the field $\mathbb{Q}$ of rational numbers is computer-practical, the field $\mathbb{A}$ of algebraic numbers is not. Short computations in $\mathbb{Q}$ have short answers. Short computations in $\mathbb{A}$ blow up exponentially.

Ready to submit?

- Please ensure all pages are in order and rotated correctly before you submit
- You will not be able to resubmit your work after the due date has passed.

Please wait...