Homework Assignment 11

Solve and submit your solutions of the following problems. Note also that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Due date

Friday, January 23, 2026 11:59 pm (Eastern Standard Time)

Late penalty

5% deducted per hour

Q1 (10 points)

Let q and p be primes in a PID R such that $p \nsim q$, let $\hat{p}$ denote the operation of "multiplication by p", acting on any R-module M, and let s and t be positive integers. On each of the R-modules R, $R/\langle q^t \rangle$, and $R/\langle p^t \rangle$, determine $\ker \hat{p}^s$ and $\operatorname{Im} \hat{p}^s$.

Q2 (15 points)

Definition. The "rank" of a module M over a commutative domain R is the maximal number of R-linearly-independent elements of M. (Linear dependence and independence is defined as in vector spaces).

Definition. An element m of a module M over a commutative domain R is called a "torsion element" if there is a non-zero $r \in R$ such that $rm = 0$. Let $\operatorname{Tor} M$ denote the set of all torsion elements of M. A module M is called a "torsion module" if $M = \operatorname{Tor} M$.

Let M be a module over a commutative domain R.

- Show that $\operatorname{Tor} M$ is always a submodule of M.
- Suppose that M has rank n and that $x_1, \ldots x_n$ is a maximal set of linearly independent elements of M. Show that $\langle x_1, \ldots x_n \rangle$ is isomorphic to R^n and that $M/\langle x_1, \ldots x_n \rangle$ is a torsion module.
- Conversely show that if M contains a submodule N which is isomorphic to R^n for some n, and so that M/N is torsion, then the rank of M is n.

Q3 (10 points)

Show that the ideal $\langle 2, x \rangle$ in $R = \mathbb{Z}[x]$, regarded as a module over R, is finitely generated but cannot be written in the form $R^k \oplus \bigoplus R/\langle p_i^{s_i} \rangle$.

Q4 (10 points)

Let M and N be modules over a ring R. In class we've defined their direct sum, $M \oplus N$. The purpose of this exercise to to give a "functional" definition of the direct sum, namely, a definition in terms of the properties that $M \oplus N$ ought to have, instead of the "constructive" definition that was given in class.

Definition An "abstract direct sum" of M and N is a triple $(S, \alpha : M \to S, \beta : N \to S)$ consisting of a module S and two morphisms α and β as indicated, such that whenever there is a triple $(P, a : M \to P, b : N \to P)$ there is a unique $\lambda : S \to P$ such that $a = \lambda \circ \alpha$ and $b = \lambda \circ \beta$.

- Prove that $M \oplus N$, along with the obvious inclusions $\alpha : M \to M \oplus N$ and $\beta : N \to M \oplus N$, is an abstract direct sum of M and N.
- Show that the abstract direct sum of M and N is unique. Namely, show that if $(S, \alpha : M \to S, \beta : N \to S)$ and $(S', \alpha' : M \to S', \beta' : N \to S')$ are both abstract direct sums of M and N, then S and S' are isomorphic.