Homework Assignment 10

Solve and submit your solutions of the following problems. Note also that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Due date

Friday, January 16, 2026 11:59 pm (Eastern Standard Time)

Late penalty

5% deducted per hour

Q1 (10 points)

In class, your Prof. embarrassed himself by showing that he doesn't know long division. Help him out by proving the lemma below and explaining how it and induction imply that if f and g are polynomials in $F[x]$ (where F is a field) and $g \neq 0$ then there exist polynomials q and r such that $f = qg + r$ and $\deg(r) < \deg(g)$.

Lemma. If f and g are polynomials in $F[x]$ and $\deg(g) \leq \deg(f)$, then there exists some polynomial (in fact, monomial) h such that $\deg(f - hg) < \deg(f)$.

(Your Prof. then further embarrassed himself by showing that he can't do subtraction of two-digit numbers. That's beyond helping; don't bother trying).

Q2 (10 points)

A commutative domain R has the property that every $x \neq 0$ in it has a unique decomposition into a product of a unit and finitely many irreducible elements (up to a permutation and up to units). Show that in R every irreducible element is a prime and therefore R is a UFD.

Hint. If x is irreducible and it divides ab, then $zx = ab$, and a, b, and ab can be written as products of irreducibles in a unique way.

Q3 (10 points)

1. Show that the ideal $I = \langle 3,\ x^3 - x^2 + 2x - 1 \rangle$ inside the ring $\mathbb{Z}[x]$ is not principal.
2. Is $\mathbb{Z}[x]/I$ a domain?

Hint for 2. Show that $\mathbb{Z}[x]/I \cong (\mathbb{Z}/3)[x] \big/ \langle x^3 - x^2 + 2x - 1 \rangle$ and that $x^3 - x^2 + 2x - 1$ is prime in the UFD $(\mathbb{Z}/3)[x]$.

Q4 (10 points)

Prove that a ring R is a PID iff it is a UFD in which $\gcd(a, b) \in \langle a, b \rangle$ for every non-zero $a, b \in R$.

Hint. Fish an element with a minimal number of factors.

Q5 (10 points)

Show that the ring $R = \mathbb{Z}[i] = \{x + iy \colon x, y \in \mathbb{Z}\} \subset \mathbb{C}$ is Euclidean and hence a PID and a UFD. What are the units of that ring?

Hint for the first part. Let $a, b \in R$ and assume $b \neq 0$. You want to find a multiple of b that's near a, relative to a norm that you are yet to find. What does the set of all multiples of b by an integer looks like? By an imaginary integer? By an element of R?

Q6 (10 points)

In $\mathbb{Z}[i]$, find the greatest common divisor of 85 and $1 + 13i$, and express it as a linear combination of these two elements.

Q7 (10 points)

Explain why $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\} \subset \mathbb{C}$ is not a UFD.

Q8 (0 points)

(Hard, for fun only) Show that the quotient ring $\mathbb{Q}[x, y]/\langle x^2 + y^2 - 1\rangle$ is not a UFD.

Ready to submit?

- Please ensure all pages are in order and rotated correctly before you submit
- You will not be able to resubmit your work after the due date has passed.

Please wait...