Homework Assignment 5

Solve and submit your solutions of the following problems. Note that the questions are not of equal values. Note also that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Due date

Saturday, October 18, 2025 11:59 pm (Eastern Daylight Time)

Late penalty

5% deducted per hour

Q1 (10 points)

Recall that if $H < G$, the *index of H in G* is $(G : H) := |G/H|$.

Let G be a group and H_1 and H_2 be finite-index subgroups of G. Show that $H_1 \cap H_2$ is also of finite index in G and that $(G : H_1 \cap H_2) \leq (G : H_1)(G : H_2)$.

Hint. Might it be true that $g(H_1 \cap H_2) = (gH_1) \cap (gH_2)$? If so, so what?

Q2 (10 points)

Let G be a group and let H be a subgroup of finite index. Prove that there is a normal subgroup N of G, contained in H, so that $(G : N)$ is also finite. (Hint: Let $(G : H) = n$ and find a morphism $G \to S_n$ whose kernel is contained in H.)

Q3 (10 points)

If p is a prime, a p-group means "a group whose order is a power of p".

Let G be a finite group and p be a prime. Show that if H is a p-subgroup of G, then $(N_G(H) : H)$ is congruent to $(G : H)$ mod p. You may wish to study the action of H on G/H by multiplication on the left.

Q4 (15 points)

For each of the following G sets X, find all of the orbits $\mathcal{O}_i$, verify that $|X| = \sum_i |\mathcal{O}_i|$, and write each orbit $\mathcal{O}_i$ as a coset space G/H_i:

1. $G = S_3$, $X = \underline{3}^2$ with $\sigma((i,j)) = (\sigma i, \sigma j)$. (Recall that $\underline{3} = \{1, 2, 3\}$).
2. $G = S_3$, $X = \underline{3}^3$ with $\sigma((i,j,k)) = (\sigma i, \sigma j, \sigma k)$.
3. $G = S_n$, $X = 2^{\underline{n}} = \mathcal{P}(\underline{n})$, with the obvious action of permutations on subsets.

Q5 (15 points)

1. Let X be a transitive G-set, let $x, y \in X$ and let $g \in G$ and assume that $gx = y$. Prove that $\mathrm{Stab}_X(x) = \mathrm{Stab}_X(y)^g$.
2. Recall that a morphism $\phi : X \to Y$ between G-sets X and Y is a function $\phi : X \to Y$ that satisfies $g\phi(x) = \phi(gx)$ for every $g \in G$ and every $x \in X$. Let X and Y be two transitive G-sets, and let $x_0 \in X$ and $y_0 \in Y$. Show that the following two statements are equivalent:

- There is a morphism $\phi : X \to Y$ of G-sets that satisfies $\phi(x_0) = y_0$.
- $\mathrm{Stab}_X(x_0) < \mathrm{Stab}_Y(y_0)$.

3. For a G-set X, let $\mathrm{Aut}(X)$ denote the set of all invertible morphisms ϕ: $X \to X$. Let X be the transitive G-set G/H, for some $H < G$. Show that for every $x, y \in X$ there is a $\phi \in \mathrm{Aut}(X)$ such that $\phi(x) = y$ if and only if H is normal in G. In other words, the action of $\mathrm{Aut}(X)$ on X is transitive iff $H \lhd X$.

Ready to submit?

- Please ensure all pages are in order and rotated correctly before you submit
- You will not be able to resubmit your work after the due date has passed.

Please wait...