This is Dror Bar-Natan's 2025-26 MAT 347 personal notebook (B). It is publically available but it comes with no guarantees whatsoever. Its content may or may not be correlated with the actual class content.

Tentative Plan.

Weeks 1-2: Ring theory. On Friday of week 2 must do modules.
Week 3: Tensor products, examples, "the ring of modules", maybe the field of fractions.
Week 4: The structure theorem of modules over a PID.
Weeks 5-11: Galois theory.
Week 12: The topological proof.

For commutative rings: R/I a field $\iff$ I is maximal

R/I a domain $\iff$ I is prime
$(ab=0 \Rightarrow a=0 \vee b=0)$ $\quad$ $ab\in I \Rightarrow a\in I \vee b\in I$

I is maximal $\Rightarrow$ I is prime

standing assumption: Our rings are commutative domains:

p prime: $(p \mid ab \Rightarrow p \mid a$ or $p \mid b) \iff (\langle p\rangle$ is prime$)$

x irreducible: $x = ab \Rightarrow a\in R^*$ or $b\in R^*$

Prime $\Rightarrow$ irreducible but irreducible $\not\Rightarrow$ prime.

in $\mathbb{Z}[\sqrt{-5}]$
2 is irred but
$2 \mid 6 = (1+\sqrt{-5})(1-\sqrt{-5})$

$\mathbb{Z}$:
Unique factorization $\rightsquigarrow$ UFD
$\Uparrow$
$\forall a,b\ \exists c\ \langle a,b\rangle = \langle c\rangle \rightsquigarrow$ PID $\quad \mathbb{Q}[x] \quad \mathbb{Z}[x]$
$\Uparrow$ $\quad \mathbb{Q}[x,y]$
$\exists$ / | remainder smaller than deno. $\rightsquigarrow$ EucD

Def A UFD (unique fact. domain) is a domain in which every non-zero element can be factored into primes: $x = u p_1 p_2 \ldots p_n$

Def PID

Def Euclidean Domain.

Thm In a UFD, factorizations are unique up to units & re-ordering.

Thm In a UFD, prime $\iff$ irreducible.

Pf If an irred. is factored, it is presented as a product of a single prime.

skip

Thm R is a UFD $\iff$ Every $x \neq 0$ has a unique decomposition into irreds.

Pf $\Rightarrow$ done.

$\Leftarrow$ need irred $\Rightarrow$ Prime. If x is irred & $x | ab$, then $zx = \underbrace{a_1 \ldots a_n}_{\text{irreds}} \underbrace{b_1 \ldots b_m}_{\text{irreds}} \Rightarrow x \sim a_i$ or $x \sim b_i \Rightarrow x | a$ or $x | b$.

Skipped

Thm In a UFD, gcd's always exist.

Def. Euclidean domain: has a "norm" $e: R - \{0\} \to \mathbb{N}$ s.t.

1. $e(ab) \geq e(a)$ 2. $\forall a, b\ \exists q, r$ s.t. $a = qb + r$ & $r = 0$ or $e(r) < e(b)$

Example. 1. $\mathbb{Z}$

2. $F[x]$

Example $\dfrac{a = x^3 - 2x^2 - 5x + 12}{b = x^2 + 1}$

$\ldots r = -6x + 14$
$a(i) = 14 - 6i$ } why?

theorem. A Euclidean domain is a "PID" (def).
(Thm: a PID is a UFD, later)

Proposition. In a PID, every prime ideal is maximal.

Pf. $I = \langle p \rangle$ prime, $I \subset J = \langle x \rangle \subset R \Rightarrow p = ax \Rightarrow$
$(a \in R^* \Rightarrow I = J) \vee (x \in R^* \Rightarrow J = R)$

Skip.

Prop In a PID, irreducible $\Rightarrow$ prime

Skipped

Pf If x is irred, $\langle x \rangle$ is maximal, for if $\langle x \rangle \subset \langle a \rangle$ then $x = ab$ & $a \in R^* \Rightarrow \langle a \rangle = R$, $b \in R^* \Rightarrow \langle a \rangle = \langle x \rangle$. So $\langle x \rangle$ is prime.

Thm PID $\Rightarrow$ UFD } done!

but first
Lemma: PID's are Noetherian } done.

Do + poly. long div. on HW.

UFD: Unique factorization domain, $x = u p_1 \ldots p_n$

PID: Principal ideal domain: I ideal $\Rightarrow I = \langle x \rangle$

Euc: $\exists\, e: R \setminus \{0\} \to \mathbb{N}$ 1. $e(ab) \geq e(a)$ 2. $\forall a, b \neq 0\ \exists\, q, r$ $a = bq + r$, $r = 0 \vee e(r) < e(b)$

Euc $\Rightarrow$ PID $\Rightarrow$ UFD

nearly! got a prod. of irreds instead of primes.

Prop In a PID, irreducible $\Rightarrow$ prime

Pf If x is irred, $\langle x \rangle$ is maximal, for if $\langle x \rangle \subset \langle a \rangle$ then $x = ab$ & $a \in R^* \Rightarrow \langle a \rangle = R$, $b \in R^* \Rightarrow \langle a \rangle = \langle x \rangle$. So $\langle x \rangle$ is prime.

Aside 1 In a UFD, irreducible $\Rightarrow$ prime } Pf: If an irred is factor, it is a product of a single prime.

Aside 2 UFD $\Leftrightarrow$ $x \neq 0$ has a unique decomposition into irreducibles } skip

gcd stuff

Finish Rings so next week can do modules?