van Kampen's thm: If $X = U_1 \cup U_2$ is a union of open sets & $U_1 \cap U_2$ is connected & $x_0 \in U_1 \cap U_2$, then

lousy version: $\pi_1(X) \cong \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2)$

better version: The obvious map

$$\pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \longrightarrow \pi_1(X)$$

is an isomorphism.

Definition: Given G_1, G_2, define

$$G_1 * G_2 = \left(\begin{array}{l} \text{words in } G_1 \sqcup G_2 = \{\bar{g} : g \in G_1\} \cup \{\hat{g} : g \in G_2\} \\ \text{under concat, mod } \bar{e} = \hat{e} = (), \ \bar{g}_1 \bar{g}_2 = \overline{g_1 g_2} \\ \quad \hat{g}_1 \hat{g}_2 = \widehat{g_1 g_2} \end{array} \right)$$

Example $\mathbb{Z} * \mathbb{Z} = \{a^n\} * \{b^n\} = F(a,b)$

$\mathbb{Z}/2 * \mathbb{Z}/2 = \dots$

Definition: Given $H \overset{\varphi_1}{\underset{\varphi_2}{\rightrightarrows}} \begin{matrix} G_1 \\ G_2 \end{matrix}$, complete it to

$$\begin{tikzcd} & G_1 \arrow[dr, "\Psi_1 : g \mapsto \bar{g}"] & \\ H \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & G_1 *_H G_2 \\ & G_2 \arrow[ur, "\Psi_2 : g \mapsto \hat{g}"'] & \end{tikzcd}$$

$$G_1 *_H G_2 := G_1 * G_2 \Big/ \begin{array}{c} \forall h \in H, \\ \overline{\varphi_1(H)} = \widehat{\varphi_2(H)} \end{array}$$

$$\begin{tikzcd} & U_1 \arrow[dr, "j_1"] & & \\ U_1 \cap U_2 \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & X = U_1 \cup U_2 \arrow[r, Rightarrow, "\pi_1"] & {} \\ & U_2 \arrow[ur, "j_2"'] & & \end{tikzcd}$$

$$\begin{tikzcd} & \pi_1(U_1) \arrow[dr] \arrow[drr, bend left, "j_{1*}"] & & \\ \pi_1(U_1 \cap U_2) \arrow[ur, "i_{1*}"] \arrow[dr, "i_{2*}"'] & & \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_2(U_2) \arrow[r] & \pi_1(U_1 \cup U_2) \\ & \pi_1(U_2) \arrow[ur, "\Psi_2"] \arrow[urr, bend right, "j_{2*}"'] & & \end{tikzcd}$$

Math 1300 Topology, Tuesday Jan 10 2006 (2 hours)

http://katlas.math.toronto.edu/0506-Topology

<u>Van-Kampen' Theorem</u> If $X = U_1 \cup U_2$, U_1 & U_2 are open $b \in U_1 \cap U_2$ and $U_1 \cap U_2$ is connected, then

in ward

$$\pi_1(X) \simeq \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2)$$

$$\begin{tikzcd} & U_1 \arrow[dr, hook, "j_1"] & \\ U_1 \cap U_2 \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & U_1 \cup U_2 \\ & U_2 \arrow[ur, hook, "j_2"'] & \end{tikzcd}$$

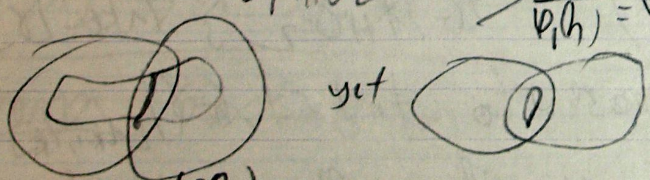
$$\Downarrow$$

$$\begin{tikzcd} & \pi_1(U_1) \arrow[dr] & \\ \pi_1(U_1 \cap U_2) \arrow[ur, "i_{1*}"] \arrow[dr, "i_{2*}"'] & & \\ & \pi_1(U_2) \arrow[ur] & \end{tikzcd}$$

If $H \xrightarrow{\varphi_1} G_1$, $H \xrightarrow{\varphi_2} G_1$ then $G_1 * G_2 = \left(\begin{array}{l} \text{Words in } G_1 \sqcup G_2 = \{\bar{g} : g \in G_1\} \dot\cup \{\hat{g} : g \in G_2\} \\ \text{under concat, mod } \bar{e} = \hat{e} = 1, \ \bar{g}_1\bar{g}_2 = \overline{g_1 g_2}, \ \hat{g}_1\hat{g}_2 = \end{array} \right.$

$$G_1 *_H G_2 = G_1 * G_2 \Big/ \overline{\varphi_1(h)} = \widehat{\varphi_2(h)} \quad \forall h \in H$$

<u>Idea</u>



yet

0. $\pi_1(S^2)$

<u>Examples</u> 1. $\pi_1(T^2)$

done line

2. $\pi_1(\Sigma_g)$ / Abelianization

4. Puncturing a 3-d nbd in X

3. $\pi_1(S^3)$ via S^3 = Union of two solid tori

5. $\pi_1(\mathbb{R}^3)$

6. $\pi_1(T_{8,3}^C)$

MAT 1301 on 250127 H10: van Kampen examples and proof.
HW03 will be online by midnight.

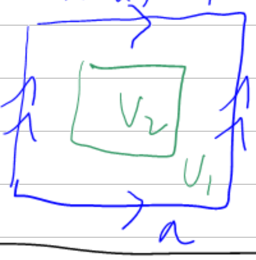
van-Kampen $X = U_1 \cup U_2$ w/ U_1, U_2 open & $U_1 \cap U_2$ connected, $x_0 \in U_1 \cap U_2$

$\Rightarrow \phi : \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \longrightarrow \pi_1(X)$ is an iso.

stronger: $X = \bigcup V_\alpha$ w/ V_α open, $x_0 \in \bigcap V_\alpha$, $V_\alpha \cap V_\beta \cap V_\gamma$ connected

$\Rightarrow \phi : \ast_{\pi_1(U_\alpha) \cap \pi_1(U_\beta)} \pi_1(U_\alpha) \longrightarrow \pi_1(X)$ is an iso.

Examples 2.



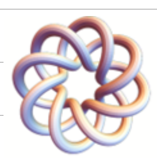
$\pi_1(T^2) = F(a,b) *_{\pi_1(S^1)} \{1\} \neq \mathbb{Z}^2$

3. $\pi_1(\Sigma_g)$ (Abelianization)

4. Puncturing a 3D manifold

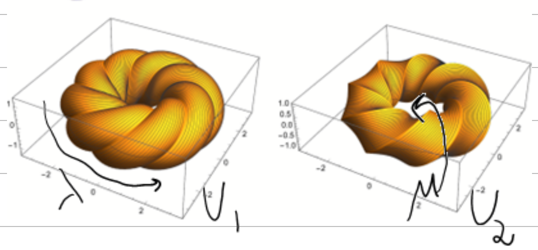
5. $\pi_1(S^3)$ as a union of two solid tori

done line

6.  $\pi_1(T_{8/3}{}^c)$

$\pi_1 = \langle \lambda \rangle *_{-} \langle \mu \rangle$

$\simeq \langle \lambda, \mu : \lambda^3 = \mu^8 \rangle$



7. Some words on Wirtinger.

Proof of easy Van Kampen 1. Construct $\Psi : [\gamma] \longrightarrow G := \pi *_{\pi} \pi$:

partition, designate, string to base, read off.

Indep of stringing, designation, partition, homotopy.

Proof of hard van-Kampen Need quad intersections!

No, with brick laying, 3 is enough.

MAT 1301 on 250128 H11-12: van Kampen examples and proof, maybe some covering spaces.
HW03 is online!

$X = U_1 \cup U_2$ w/ U_1, U_2 open & $U_1 \cap U_2$ connected, $x_0 \in U_1 \cap U_2$

$\Rightarrow \phi : \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) \longrightarrow \pi_1(X)$ is an iso.

$X = \bigcup U_\alpha$ w/ U_α open, $x_0 \in \bigcap U_\alpha$, $U_\alpha \cap U_\beta$ & $U_\alpha \cap U_\beta \cap U_\gamma$ connected

$\Rightarrow \phi : \ast_{\pi_1(U_\alpha)\cap\pi_1(U_\beta)} \pi_1(U_\alpha) \longrightarrow \pi_1(X)$ is an iso.

S^3 is a union of two solid tori.

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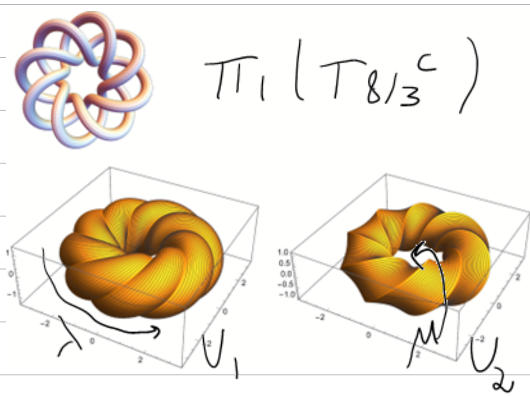
7. 

$\pi_1 = \langle \lambda \rangle *_{-} \langle \mu \rangle$

$\simeq \langle \lambda, \mu : \lambda^3 = \mu^8 \rangle$

8. Some words on Wirtinger.

<u>Proof of ^easy^ Van Kampen</u>: Construct $\Psi : [\gamma] \longrightarrow G := \pi *_\pi \pi$:

partition, designate, string to base, read off.

Indep of stringing, designation, partition, homotopy.

<u>proof of hard van Kampen</u> Need quad intersections?

No, with brick laying, 3 is enough.

Pushout:

$$\begin{tikzcd} & B_1 \arrow[dr, "\psi_1"] \arrow[drr, bend left] & & \\ A \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] & & C \arrow[r, dashed, "\exists\theta"] & D \\ & B_2 \arrow[ur, "\psi_2"'] \arrow[urr, bend right] & & \end{tikzcd}$$

IF C exists, it is unique up to iso

Examples

$$\begin{tikzcd} & U_1 \\ U_1 \cap U_2 \arrow[ur] \arrow[dr] & & X = U_1 \cup U_2 \quad Y \\ & U_2 \end{tikzcd}$$

w/ some small print, π_1 takes pushouts to pushouts. $\Leftarrow$

$$\begin{tikzcd} & G_1 \\ H \arrow[ur] \arrow[dr] & & G_1 *_H G_2 \quad K \\ & G_2 \end{tikzcd}$$