Homework Assignment 6

Problem 1. Just in case you thought commutative diagrams cannot get any worse, here's a question to prove you wrong (though actually, it is surprisingly easy).

1. Define a category $\mathcal{S}$ of "short exact sequences of chain complexes".
2. Define a category $\mathcal{L}$ of "long exact sequences".
3. Construct a functor $\mathcal{H}\colon \mathcal{S} \to \mathcal{L}$. (We've constructed this functor on objects already, so you don't need to do that again. The challenge is to do it on morphisms and to verify that starting from a morphism $\mathcal{F}$ in $\mathcal{S}$, its image $\mathcal{H}(\mathcal{F})$ is indeed a morphism in $\mathcal{L}$).

Problem 2. Given a morphism $f\colon (X,A) \to (Y,B)$, show that the diagram

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{\delta} & H_{n-1}(A) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ H_n(Y,B) & \xrightarrow{\delta} & H_{n-1}(B) \end{array}$$

is commutative.

Problem 3. Given a triple of spaces $B \subset A \subset X$, construct a long exact sequence relating $H_*(X,A)$, $H_*(A,B)$, and $H_*(X,B)$.

Problem 4. Khovanov homology is an invariant of knots obtained by first defining a chain complex $\mathcal{C}$, and then taking its homology. To show that Khovanov homology does not change when the knot moves, one has to show that pieces of $\mathcal{C}$ can be cancelled off — be removed without changing the homology.

1. Suppose $\mathcal{C}'$ is an acyclic subcomplex of $\mathcal{C}$ ("acyclic" is a different word for "exact", or "having no homology"). Show that $\mathcal{C}'$ can be cancelled off. Namely, that the quotient $\mathcal{C}/\mathcal{C}'$ has the same homology as the original complex $\mathcal{C}$.
2. Suppose $\mathcal{C}'$ is a subcomplex of $\mathcal{C}$, and that the quotient complex $\mathcal{C}/\mathcal{C}'$ is acyclic. Then $\mathcal{C}'$ has the same homology as $\mathcal{C}$ (roughly, the complement of $\mathcal{C}'$ can be cancelled off).
3. There is a third theorem along these lines. What does it say?