Homework Assignment 8

Nov 17 Q2 a) Prove $\sim$ is an equivalence relation

Pf: Let $F_0: X \to Y$, $F_1: X \to Y$, $F_2: X \to Y$ be continuous functions

① Show $F_0 \sim F_0$: Define $H: X \times I \to Y$ by $H(x,t) = F_0(x)$, then H is continuous as F_0 is, $\forall x \in X$, $H(x,0) = F_0(x)$, $H(x,1) = F_0(x)$. Hence F_0 and F_0 are homotopic.

② Show $F_0 \overset{H'}{\sim} F_1 \Rightarrow F_1 \overset{H}{\sim} F_0$: Assume $F_0 \sim F_1$ then there exists a continuous $H': X \times I \to Y$ s.t. $H'(x,0) = F_0(x)$, $H'(x,1) = F_1(x)$. Define $H: X \times I \to Y$ by $H(x,t) = H'(x,1-t)$. Since H' is continuous, then H is continuous. $\forall x \in X$, $H(x,0) = H'(x,1) = F_1(x)$, $H(x,1) = H'(x,0) = F_0(x)$, hence F_1 and F_0 are homotopic.

③ Show $F_0 \overset{H'}{\sim} F_1$, $F_1 \overset{H''}{\sim} F_2 \Rightarrow F_0 \overset{H}{\sim} F_2$: Assume $F_0 \sim F_1$ and $F_1 \sim F_2$, then there exists continuous $H': X \times I \to Y$ and $H'': X \times I \to Y$ s.t. $\forall x \in X$, $H'(x,0) = F_0(x)$, $H'(x,1) = F_1(x)$, $H''(x,0) = F_1(x)$, $H''(x,1) = F_2(x)$. Now define $H: X \times I \to Y$ as follows:

$$H(x,t) = \begin{cases} H'(x,2t), & 0 \le t \le \frac{1}{2} \\ H''(x,2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

H is well-defined: if $t = \frac{1}{2}$, $H'(x,1) = F_1(x) = H''(x,0)$. Because H' is continuous on the closed subset $X \times [0,\frac{1}{2}]$ and H'' is continuous on the closed subset $X \times [\frac{1}{2},1]$, H is continuous on $X \times I$. Also, $\forall x \in X$, $H(x,0) = H'(x,0) = F_0(x)$, $H(x,1) = H''(x,1) = F_2(x)$. Thus $F_0 \sim F_2$.

$\blacksquare$

b) If γ is a path in X and $F: X \to Y$ is continuous, $F_* \gamma = F \circ \gamma$ is a path in Y.
Show if $\gamma_0 \underset{p}{\sim} \gamma_1$ in X, then $F_* \gamma_0 \underset{p}{\sim} F_* \gamma_1$ in Y.

Pf: Assume $\gamma_0 \underset{p}{\sim} \gamma_1$, then they have the same start point x_0 and endpoint x_1, and there exists a continuous $H': I \times I \to X$ s.t. $H'(s,0) = \gamma_0(s)$, $H'(s,1) = \gamma_1(s)$, $H'(0,t) = x_0$, $H'(1,t) = x_1$

- Define $H: I \times I \to Y$ by $H(s,t) = F(H'(s,t))$. Since F and H' are continuous then $H = F \circ H'$ is continuous
- Show $F_* \gamma_0 \overset{H}{\underset{p}{\sim}} F_* \gamma_1$: $H(s,0) = F(H'(s,0)) = F(\gamma_0(s)) = F_* \gamma_0(s)$
  $H(s,1) = F(H'(s,1)) = F(\gamma_1(s)) = F_* \gamma_1(s)$
  $H(0,t) = F(H'(0,t)) = F(x_0)$, the starting point of $F_* \gamma_0$ and $F_* \gamma_1$
  $H(1,t) = F(H'(1,t)) = F(x_1)$, the end point of $F_* \gamma_0$ and $F_* \gamma_1$

Hence $F_* \gamma_0 \underset{p}{\sim} F_* \gamma_1$, as needed.

$\blacksquare$

c) Prove if $F_i : X \to Y$, $G_i : Y \to Z$ are continuous for $i = 0,1$, and if $F_0 \sim F_1$, $G_0 \sim G_1$, then $G_0 \circ F_0 \sim G_1 \circ F_1$.

Pf: Assume $F_0 \sim F_1$, $G_0 \sim G_1$, then there exists continuous $H' : X \times I \to Y$ and $H'' : Y \times I \to Z$ s.t. $\forall x \in X$, $H'(x,0) = F_0(x)$, $H'(x,1) = F_1(x)$; $\forall y \in Y$, $H''(y,0) = G_0(y)$, $H''(y,1) = G_1(y)$.

- Define $H : X \times I \to Z$ by $H(x,t) = H''(H'(x,t),t)$
- Since H' and H'' are continuous, then H is continuous
- $\forall x \in X$, $H(x,0) = H''(H'(x,0),0) = H''(F_0(x),0) = G_0(F_0(x))$

  $H(x,1) = H''(H'(x,1),1) = H''(F_1(x),1) = G_1(F_1(x))$

Thus $G_0 \circ F_0 \overset{H}{\sim} G_1 \circ F_1$, as needed.

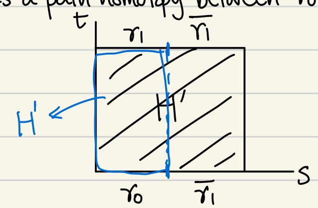
$\blacksquare$

Q3 Let X be a path connected space. X is simply connected if for some $x_0 \in X$, the group $\pi_1(X, x_0)$ is trivial. $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ is the unit circle in $\mathbb{R}^2$

a) Show X is simply connected iff any paths γ_0 and γ_1 that have the same end points in X are path homotopic.

Pf: $\Rightarrow$: Assume X is simply connected and let 2 paths γ_0 and γ_1 be given s.t. $\gamma_0(0) = \gamma_1(0) = x_0$ and $\gamma_0(1) = \gamma_1(1) = x_1$. Show $\gamma_0 \underset{p}{\simeq} \gamma_1$

- Consider $\pi(X, x_0)$. by X is path connected and simply connected, the foundamental group for any base point is trivial. Hence $\pi(X, x_0) = \{[e_{x_0}]\}$.
- Since γ_0 and γ_1 have the same endpoints, then $\gamma_0 * \overline{\gamma_1} \in \pi(X, x_0) \Rightarrow \gamma_0 * \overline{\gamma_1} \in [e_{x_0}]$. Similarly, $\gamma_1 * \overline{\gamma_1} \in [e_{x_0}]$. By the equivalence relation of $\sim$, $\gamma_0 * \overline{\gamma_1} \underset{p}{\simeq} e_{x_0} \underset{p}{\simeq} \gamma_1 * \overline{\gamma_1}$ $\Rightarrow \gamma_0 * \overline{\gamma_1} \underset{p}{\simeq} \gamma_1 * \overline{\gamma_1}$. Then there exists a path homotopy $H' : I \times I \to X$ between $\gamma_0 * \overline{\gamma_1}$ and $\gamma_1 * \overline{\gamma_1}$.
- H' induces a path homotopy between γ_0 and γ_1 as follows:



then $\gamma_0 \underset{p}{\simeq} \gamma_1$, as needed.

$\Leftarrow$: Assume any two paths γ_0 and γ_1 with the same endpoints in X are path-homotopic. Show X is simply connected.

- wlog assume X is non-empty so can take $x_0 \in X$ and show $\pi(X, x_0)$ is trivial.
- Fix path $\gamma : [0,1] \to X$ in $\pi(X, x_0)$, so $\gamma(0) = x_0$ and $\gamma(1) = x_0$. Show $\gamma \in [e_{x_0}]$, which is to show $\gamma \underset{p}{\simeq} e_{x_0}$. Wlog suppose $\gamma \neq e_{x_0}$, otherwise by the reflexivity of "$\sim$" we're done. Then $\exists\, p_0 \in (0,1)$ s.t. $\gamma(p_0) \neq x_0$. Let $\gamma(p_0) = x_1$.
- Then $\gamma|[0,p_0]$ and $\overline{\gamma|[p_0,1]}$ are 2 paths from x_0 to x_1. By assumption, $\gamma|[0,p_0] \underset{p}{\simeq} \overline{\gamma|[p_0,1]}$. So there exists a path homotopy H' from $\gamma|[0,p_0]$ to $\overline{\gamma|[p_0,1]}$.
- Define $H : I \times I \to X$ by $H(s,t) = \begin{cases} H'(s,t), & \text{if } 0 \le s \le p_0 \\ \gamma(s), & \text{otherwise} \end{cases}$. Then H is a homotopy between $\gamma|[0,p_0] * \gamma|[p_0,1]$ and $\overline{\gamma|[p_0,1]} * \gamma|[p_0,1]$, but then this is $\gamma \overset{H}{\underset{p}{\simeq}} e_{x_0}$.

∎

b) Show X is simply connected iff every continuous function $\lambda: S^1 \to X$ is homotopic to a constant function.

<u>Pf:</u> $\Rightarrow$: Assume X is simply connected. Fix continuous $\lambda: S^1 \to X$ and show $\exists\, x_0 \in X$ and constant function $F_{x_0}: S^1 \to X$ s.t. $F_{x_0} = x_0$ and $\lambda \sim F_{x_0}$.

- We can parametrize S^1 into a path $\gamma: [0,1] \to \mathbb{R}^2$ by $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$, then γ starts and ends at $(1,0)$. Then $\lambda \circ \gamma: [0,1] \to X$ is a path in X that starts and ends at $\lambda(1,0)$, let $x_0 = \lambda(1,0)$, show $\lambda \sim F_{x_0}$
- Notice both $\lambda \circ \gamma$ and e_{x_0} are paths of the same endpoints, by X is simply connected and Q3a, $\lambda \circ \gamma \overset{H'}{\underset{p}{\simeq}} e_{x_0}$, where H' is a path homotopy between $\lambda \circ \gamma$ and e_{x_0}
- Define $H: S^1 \times I \to X$ by

$$H((u,v),t) = \begin{cases} H'(\arctan(\frac{v}{u}), t), & u>0 \\ H'(\pi + \arctan(\frac{v}{u}), t), & u<0 \\ H'(\frac{1}{4}, t), & \text{if } (u,v) = (0,1) \\ H'(\frac{3}{4}, t), & \text{if } (u,v) = (0,-1). \end{cases}$$

By checking the limit of H at the ends of each subinterval in the definition above, H can be shown continuous. (And at the interior of each subinterval H is continuous since H' is.)

$$H((u,v),0) = \left.\begin{cases} H'(\arctan(\frac{v}{u}), 0) = \lambda \circ \gamma(\arctan(\frac{v}{u})) = \lambda(u,v), & u>0. \\ H'(\pi + \arctan(\frac{v}{u}, 0)) = \lambda \circ \gamma(\pi + \arctan(\frac{v}{u})) = \lambda(u,v), & u<0 \\ H'(\frac{1}{4}, 0) = \lambda \circ \gamma(\frac{1}{4}) = \lambda(0,1), & \text{if } (u,v) = (0,1) \\ H'(\frac{3}{4}, t) = \lambda \circ \gamma(\frac{3}{4}) = \lambda(0,-1), & \text{if } (u,v) = (0,-1) \end{cases}\right\} \text{either way, } H((u,v),0) = \lambda(u,v).$$

In similar manner, we can check $H((u,v),1) = e_{x_0}(u,v) = x_0 = F_{x_0}(u,v)$.

Thus H is a homotopy between λ and the constant function F_{x_0}, as needed.

$\Rightarrow$: See next page

$\Leftarrow$: Assume every continuous $\lambda: S^1 \to X$ is homotopic to a constant function. Show X is simply connected.

- Wlog X is non-empty, so can pick $x_0 \in X$ and we show $\pi(X, x_0) = \{[e_{x_0}]\}$. Fix a loop γ in $\pi(X, x_0)$ and we can parametrize it by some $\hat{\lambda}: [0,1] \to X$ s.t. $\hat{\lambda}(0) = \hat{\lambda}(1) = x_0$. Notice that as $\hat{\lambda}(0) = \hat{\lambda}(1)$, the domain of $\hat{\lambda}$ is $[0,1] \setminus 0 \sim 1 \cong S^1$. Then $\hat{\lambda}$ can also be considered as a continuous function from S^1 to X, which by assumption, is homotopic to some constant function e_{x_1}. In other words, $\hat{\lambda} \overset{H'}{\simeq} e_{x_1}$, where $H': I \times I \to X$ is the homotopy between them.
- It follows that $H'(0,t)$ is a path that traces x_0 to x_1 in the retraction of $\hat{\lambda}$ to e_{x_1}. Let's call this path ϕ. Then $\phi * \bar{\phi}$ is a loop at x_0 that sets out to x_1 and returns back to x_0 following the same path. By a theorem, $\phi * \bar{\phi} \simeq_p e_{x_0}$.

Thus, it suffices to show $\gamma \simeq_p \phi * \bar{\phi}$, then by transitivity of $\sim$, we will have $\gamma \simeq_p e_{x_0}$, as desired.

- Define $H: I \times I \to X$ by:

$$H(s,t) = \begin{cases} H'(0, 3st), & 0 \le s \le \frac{1}{3} \\ H'(3s-1, t), & \frac{1}{3} \le s \le \frac{2}{3} \\ H'(0, 3t(1-s)), & \frac{2}{3} \le s \le 1 \end{cases}$$

We show H is a path-homotopy between γ and e_{x_0}:

(1) H is continuous: since H' is continuous, then it suffices to check continuity of H at its subinterval endpoints. Indeed, $s = \frac{1}{3}$: $H'(0, 3st) = H'(0,t)$, $H'(3s-1, t) = H'(0,t)$, $s = \frac{2}{3}$: $H'(3s-1, t) = H'(1,t)$, we remember that at any t, $H'(1,t)$ is a retraction of the loop γ, hence it is also a loop, so $H'(1,t) = H'(0,t)$. But then this agrees with $H'(0, 3t(1-\frac{2}{3}))$ $= H'(0,t)$, so H is continous at $s = \frac{2}{3}$ too.

(2) $H(0,t) = H'(0,0) = \hat{\lambda}(0) = x_0$ $\qquad H(1,0) = H'(0,0) = x_0$

(3) $H(s,0) = H'(0,0) * H'(3s-1, 0)\big|_{\frac{1}{3} \le s \le \frac{2}{3}} * H'(0,0)$

$= x_0 * H'(3s-1, 0)\big|_{\frac{1}{3} \le s \le \frac{2}{3}} * x_0$

$\simeq_p H'(3s-1, 0)\big|_{\frac{1}{3} \le s \le \frac{2}{3}}$, by a change of variable this is $H'(\tilde{s}, 0)$ where $0 \le \tilde{s} \le 1$, and by definition of H', this is $\hat{\lambda}(\tilde{s})$ for $0 \le \tilde{s} \le 1$, which is precisely γ.

$H(s,1) = H'(0, 3s)\big|_{0 \le s \le \frac{1}{3}} * H'(3s-1, 1)\big|_{\frac{1}{3} \le s \le \frac{2}{3}} * H'(0, 3(1-s))\big|_{\frac{2}{3} \le s \le 1}$

$= \underbrace{H'(0, 3s)\big|_{0 \le s \le \frac{1}{3}}}_{①} * e_{x_1}(3s-1) * \underbrace{H'(0, 3-3s)\big|_{\frac{2}{3} \le s \le 1}}_{②}$

$= ① * x_1 * ②$

①: By a change of variable, is equivalent to $H'(0,\tilde{s})$ for $0\le\tilde{s}\le 1$, which is ϕ.

②: Similarly, it is equivalent to $H'(0,1-\tilde{s})$ for $0\le\tilde{s}\le 1$, which is $\bar{\phi}$

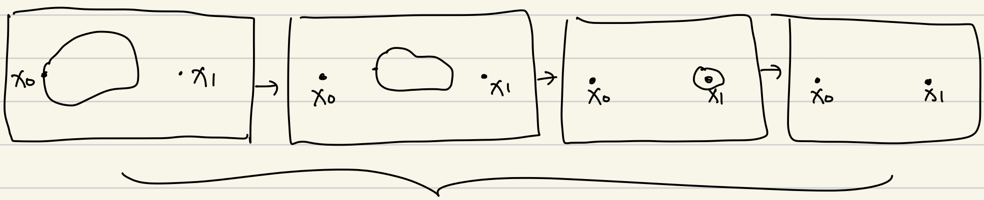
Thus $H(s,1) = \phi * x_1 * \bar{\phi} = \phi * \bar{\phi}$

- Since we have verified $H(0,t)=x_0$, $H(1,t)=x_0$, $H(s,0)=\gamma$, and $H(s,1)=\phi*\bar{\phi}$, $\gamma \simeq_p \phi*\bar{\phi}$. This completes the proof.

$\blacksquare$

Below is a visual demonstration:

- under the homotopy H' between the parametrizing function $\hat{\lambda}$ of γ and e_{x_1}:

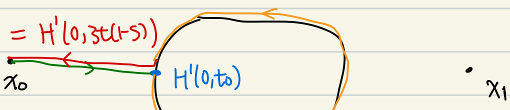


as t progresses in $H'(s,t)$.

- H is defined correspondingly. At $t=t_0$:

$\frac{2}{3}\le s\le 1$:

$H(s,t) = H'(0,3t(1-s))$

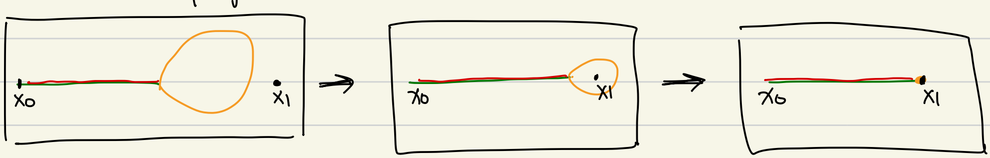


$0\le s\le\frac{1}{3}$:

$H(s,t) = H'(0,3st_0)$

$\frac{1}{3}\le s\le\frac{2}{3}$: $H(s,t)=H'(3s-1,t_0)$

- Then as t progresses:



Warning: The author is not confident of their solution of Q4.

Q4 $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ S^2 with 1 point removed is homeomorphic to $\mathbb{R}^2$

a) Show any continuous $\lambda_0 : S^1 \to S^2$ which is not surjective is homotopic to a constant function.

Pf: • Suppose continuous $\lambda_0 : S^1 \to S^2$ is not surjective, then $\exists p \in S^2$ s.t. $\lambda_0(S^1) \subset S^2 \setminus \{p\}$. Then $S^2 \setminus \{p\}$ is homeomorphic to $\mathbb{R}^2$: $\exists$ homeomorphism $f : S^2 \setminus \{p\} \to \mathbb{R}^2$, then f^{-1} also exists and is continuous

• $f \circ \lambda_0 : S^1 \to \mathbb{R}^2$, since $\mathbb{R}^2$ is simply connected, by Q3b, $\exists q \in S^1$ and constant function $e_{f(\lambda_0(q))}$ s.t. $f \circ \lambda_0 \sim e_{f(\lambda_0(q))}$, let H' be their homotopy.

• show $\lambda_0 \sim e_{\lambda_0(q)}$: let $H : S^1 \times I \to S^2$ be $H(s,t) = f^{-1}(H'(s,t))$. H is continuous as f^{-1} & H' are. $H(s,0) = f^{-1}(H'(s,0)) = f^{-1}(f \circ \lambda_0(s)) = \lambda_0(s)$.

$$H(s,1) = f^{-1}(H'(s,1)) = f^{-1}(e_{f(\lambda_0(q))}(s)) = f^{-1}(f(\lambda_0(q))) = \lambda_0(q) = e_{\lambda_0(q)}(s).$$

Thus λ_0 is homotopic to a constant function.

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b) Show any continuous $\lambda : S^1 \to S^2$ is homotopic to a continuous non-surjective $\lambda_0 : S^1 \to S^2$.

Pf: It suffices to consider a surjective continuous $\lambda : S^1 \to S^2$ and we show it is homotopic to a non-surjective continuous function from S^1 to S^2. Since $S^1 \cong [0,1]/0\sim1$, then it is equivalent to consider $\hat{\lambda} : [0,1] \setminus 0\sim1 \to S^2$, a closed path that is surjective on S^2, induced by λ.

• Let $\varepsilon > 0$ be small and $\mathcal{U}$ be the collection of open disks of radius ε on S^2.

Since S^2 is compact, by the Lebesgue Number Lemma, $\exists \delta > 0$ s.t. $\forall p \in S^2$, $\exists U \in \mathcal{U}$ s.t. $B_\delta(p) \subset U$. Then $\{\hat{\lambda}^{-1}(B_\delta(p)) : p \in S^2\}$ is an open cover of $[0,1]$.

• $[0,1]$ is compact so there is a finite subcover $\{\hat{\lambda}^{-1}(B_\delta(p_i)) : 1 \le i \le n\} = \{(a_i, a_{i+1}) : 1 \le i \le n\}$ of $[0,1]$. For any i, $\lambda((a_i, a_{i+1})) \subset B_\delta(p_i) \subset U$ for some ε-disk $U \in \mathcal{U}$, by the Lebesgue Number Lemma. So $\lambda((a_i, a_{i+1})) \subset U$. Since ε is small, it is ensured that $\lambda((a_i, a_{i+1}))$ is a single line segment in U.

• We know U is homeomorphic to $\mathbb{R}^2$ and $\mathbb{R}^2$ is simply connected, any paths in U with the same endpoints are path-homotopic. Then, $\lambda((a_i, a_{i+1}))$ is path-homotopic to a straight line segment in U with the same endpoints.

• By concatenating path-homotopic paths, we get the image of $\hat{\lambda} = \hat{\lambda}|_{(a_1,a_2)} * \cdots * \hat{\lambda}|_{(a_n,a_{n+1})}$ being path-homotopic to the finite concatenation of line segments, which is a line segment. Then $\hat{\lambda}$ is homotopic to a continuous function that maps S^1 onto a line segment in S^2, and this, of course, cannot be surjective in S^2.

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c) With the same language as the previous exercise, deduce S^2 is simply connected.

By Q3b), S^2 is simply connected iff any continuous $\lambda : S^1 \to S^2$ is homotopic to a constant function. By Q4b), any continuous $\lambda : S^1 \to S^2$ is homotopic to a non-surjective $\lambda_0 : S^1 \to S^2$, by Q4a) a non-surjective $\lambda_0 : S^1 \to S^2$ is homotopic to a constant function. So by transitivity of '$\sim$' (proven in Q2a), any continuous $\lambda : S^1 \to S^2$ is homotopic to a constant function, thus S^2 is simply connected.

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