MAT327 – HW5

م

2024-10-16

Question 2

Let $x_1, x_2, \ldots$ be a sequence of points in a product space $\prod X_\alpha$. Show that this sequence converges to the point x iff $\pi_\alpha(x_k) \to \pi_\alpha(x)$ for every α. Is the same fact true in the box topology?

Answer.

($\Longrightarrow$) Suppose the sequence $(x_k)_{k=1}^{\infty}$ converges to x. Let α be arbitrary and $U \subset X_\alpha$ be an open neighborhood of $\pi_\alpha(x)$. Then $\pi_\alpha^{-1}(U)$ is an open neighborhood of x, so there exists $N \in \mathbb{N}$ such that $k > N$ implies $x_k \in \pi_\alpha^{-1}(U)$. Then, if $k > N$, we have $\pi_\alpha(x_k) \in \pi_\alpha(\pi_\alpha^{-1}(U)) \subset U$. That is, if $k > n$, then $\pi_\alpha(x_k) \in U$. Thus, for every open neighborhood of $\pi_\alpha(x)$, the sequence $\pi_\alpha(x_k)$ eventually resides in said neighborhood, so $\pi_\alpha(x_k)$ converges to $\pi_\alpha(x)$, for every α.

($\Longleftarrow$) Suppose the sequence $\pi_\alpha(x_k)$ converges to $\pi_\alpha(x)$ for every α. Let U be an open neighborhood of x and let $B \subset U$ be a basic set containing x. Then, for some $\alpha_1, \ldots, \alpha_m$, and $U_{\alpha_1}, \ldots, U_{\alpha_m}$, we have

$$B = \pi_{\alpha_1}^{-1}\left(U_{\alpha_1}\right) \cap \ldots \cap \pi_{\alpha_m}^{-1}\left(U_{\alpha_m}\right)$$

In particular, since $x \in \pi_{\alpha_i}^{-1}\left(U_{\alpha_i}\right)$, we have $\pi_{\alpha_i}(x) \in U_{\alpha_i}$, for each $1 \leq i \leq m$. Since the sequence $\pi_{\alpha_i}(x_k)$ converges to $\pi_{\alpha_i}(x)$, we thus have for each $1 \leq i \leq m$, some $N_i \in \mathbb{N}$ such that $n > N_i$ implies $\pi_{\alpha_i}(x_n) \in U_{\alpha_i}$. We set $N = \max_{1 \leq i \leq m} N_i$. Then, if $n > N$, we have for every $1 \leq i \leq m$, $\pi_{\alpha_i}(x_n) \in U_{\alpha_i}$, which is to say

$$x_n \in \pi_{\alpha_1}^{-1}\left(U_{\alpha_1}\right) \cap \ldots \cap \pi_{\alpha_m}^{-1}\left(U_{\alpha_m}\right) = B$$

Thus, if $n > N$, $x_n \in B \subset U$. Therefore, for every open neighborhood of x, the sequence (x_k) eventually resides in said neighborhood, so (x_k) converges to x.

Both implications demonstrate that the sequence (x_k) converges to x if and only if the sequence $(\pi_\alpha(x_k))$ converges to $\pi_\alpha(x)$ for every α. ■

No, the same fact is not true in the box topology. While convergence of the sequence (x_k) does imply the convergence of the sequence $\pi_\alpha(x_k)$ to $\pi_\alpha(x)$ for every α (the same proof works unchanged), the reverse implication no longer holds.

To see why, consider the product space $\prod_{i \in \mathbb{N}} \mathbb{R}$ and the sequence

$$x_k = (\underbrace{1,\ 1,\ \ldots,\ 1}_{k \text{ times}}, 0, 0, \ldots,)$$

That is, x_k is the sequence where the first k elements are 1, and the remaining are 0.

Let $x \in \prod_{i \in \mathbb{N}} \mathbb{R}$ be the sequence which is constantly 1, $x = (1, 1, 1, \ldots)$. It is easy to see that $\pi_i(x_k)$ converges to $\pi_i(x)$ for every $i \in \mathbb{N}$, for the sequence $\pi_i(x_k)$ is constantly 1 after the first i terms.

On the other hand, it is impossible for the sequence (x_k) to converge to x. Consider the neighborhood

$$U = \prod_{i \in \mathbb{N}} \left(\frac{1}{2}, \frac{3}{2} \right)$$

which is open in the box topology and contains x. For any $N \in \mathbb{N}$, the sequence element $x_{N+1} \notin U$, for it's $(N+2)$th element is zero and therefore not in $\left(\frac{1}{2}, \frac{3}{2}\right)$. Thus, it's impossible for any tail of the sequence to reside in U, so (x_k) can not converge to x in the box topology.

Question 3

Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\mathbb{N}$ consisting of the sequences that are almost always 0 - meaning, that are not zero only for finitely many indices. What is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\mathbb{N}$ in the box and in the cylinders topology?

Answer. The closure of $\mathbb{R}^\infty$ is $\mathbb{R}^\infty$ in the box topology and $\mathbb{R}^\mathbb{N}$ in the cylinders topology.

Closure in the box topology. We show that if $x \in \mathbb{R}^\mathbb{N}$ is not in $\mathbb{R}^\infty$, then $x \notin \overline{\mathbb{R}^\infty}$.

Let $x \in \mathbb{R}^\mathbb{N} \setminus \mathbb{R}^\infty$. For each $i \in \mathbb{N}$, define

$$U_i = \begin{cases} (-1, 1) & x_i = 0 \\ \left(x_i - \frac{|x_i|}{2}, x_i + \frac{|x_i|}{2}\right) & \text{otherwise} \end{cases}$$

and set

$$U = \prod_{i \in \mathbb{N}} U_i$$

Then, U is an open neighborhood of x, since each U_i is open and $x_i \in U_i$ for every i by construction. We claim that U can not intersect $\mathbb{R}^\infty$. Let $y \in U$. Since $x \notin \mathbb{R}^\infty$, there is an infinite sequence $k_1, k_2, \ldots$ such that $x_{k_i} \neq 0$. For each such k_i, we have

$$y_{k_i} \in U_{k_i} = \left(x_{k_i} - \frac{|x_{k_i}|}{2}, x_{k_i} + \frac{|x_{k_i}|}{2}\right)$$

which necessitates $y_{k_i} \neq 0$. Since y is non-zero at infinitely many points, it can't be in $\mathbb{R}^\infty$. Thus, U is an open neighborhood of x not intersecting $\mathbb{R}^\infty$, so $x \notin \overline{R^\infty}$.

Therefore, $\overline{R^\infty} = \mathbb{R}^\infty$. ∎

Closure in the cylinders topology. We show that the entirety of $\mathbb{R}^\mathbb{N}$ is in $\overline{\mathbb{R}^\infty}$.

Let $x \in \mathbb{R}^\mathbb{N}$ and B be a basic neighborhood of x. Then, for some $i_1, \ldots, i_k \in \mathbb{N}$ and open sets $U_{i_1}, \ldots, U_{i_k} \subset \mathbb{R}$, we have

$$B = \pi_{i_1}^{-1}\left(U_{i_1}\right) \cap \ldots \cap \pi_{i_k}^{-1}\left(U_{i_k}\right)$$

Define $y \in \mathbb{R}^\mathbb{N}$ by

$$y_i = \begin{cases} x_i & i \in \{i_1, \ldots, i_k\} \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $y \in \mathbb{R}^\infty$, for y_i can be non-zero only if $i \in \{i_1, \ldots, i_k\}$ which is a finite set. Also, $y \in B$, for $\pi_i(y) = x_i \in U_i$ for each $i \in \{i_1, \ldots, i_k\}$.

Thus, every open neighborhood of x intersects $\mathbb{R}^\infty$, so $x \in \overline{\mathbb{R}^\infty}$ and hence $\overline{\mathbb{R}^\infty} = \mathbb{R}^\mathbb{N}$, as needed. ∎

Question 4

Show that $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology is metrizable.

Answer. Recall the result of Question 4 in HW3, where we showed that the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ was in fact equal to the product topology on $\mathbb{R}_d \times \mathbb{R}$, where $\mathbb{R}_d$ is $\mathbb{R}$ with the discrete topology. Thus, it suffices to show that the product of metric topologies is metrizable, which we do in the following lemma.

Lemma. Let X, Y be metric spaes with metrics d_X, d_Y respectively. Then, the product topology on $X \times Y$ is metrizable.

Proof. Define $d : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$$

Then, d is always non-negative, for d_X, d_Y are always non-negative, and

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) = 0 &\iff d_X(x_1, x_2) = 0 \text{ and } d_Y(y_1, y_2) = 0 \\ &\iff x_1 = x_2 \text{ and } y_1 = y_2 \\ &\iff (x_1, y_1) = (x_2, y_2) \end{aligned}$$

Similarly, since d_X, d_Y are symmetric,

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) &= d_X(x_1, x_2) + d_Y(y_1, y_2) \\ &= d_X(x_2, x_1) + d_Y(y_2, y_1) \\ &= d((x_2, y_2), (x_1, y_1)) \end{aligned}$$

so d is symmetric. Finally, by the triangle inequality on d_X, d_Y, we have

$$\begin{aligned} d((x_1, y_1), (x_3, y_3)) &= d_X(x_1, x_3) + d_Y(y_1, y_3) \\ &\le d_X(x_1, x_2) + d_X(x_2, x_3) + d_Y(y_1, y_2) + d_Y(y_2, y_3) \\ &= d_X(x_1, x_2) + d_Y(y_1, y_2) + d_X(x_2, x_3) + d_Y(y_2, y_3) \\ &= d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) \end{aligned}$$

so d satisfies the triangle inequality.

Finally, we show that the metric topology induced by d is equal to the product topology. Let $(p, q) \in X \times Y$.

Suppose B be a basic set of the product topology containing (p, q). Then, we have $r, s > 0$ such that $B_r(p) \times B_s(q) \subset B$. Setting $t = \min(r, s)$, we claim that $B_t((p, q)) \subset B_r(p) \times B_s(q)$. Indeed, if $(u, v) \in B_t((p, q))$, then we have

$$d_X(u, p) + d_Y(v, q) < t$$

so, since both d_X, d_Y are non-negative, we have both $d_X(u, p) < t \le r$ and $d_Y(v, q) < t \le s$ and hence $(u, v) \in B_r(p) \times B_s(q)$.

Conversely, suppose B is a basic set of the metric topology induced by d containing (p, q). Then, we have $\varepsilon > 0$ such that $B_\varepsilon((p, q)) \subset B$. We claim that $B_{\varepsilon/2}(p) \times B_{\varepsilon/2}(q) \subset B_\varepsilon((p, q))$. Indeed, if $(u, v) \in B_{\varepsilon/2}(p) \times B_{\varepsilon/2}(q)$, then $d_X(u, p) < \frac{\varepsilon}{2}$ and $d_Y(v, q) < \frac{\varepsilon}{2}$, so

$$d((u, v), (p, q)) = d_X(u, p) + d_Y(v, q) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, the metric topology induced by d and the product topology of X, Y are each finer than the other, so they're equal. Therefore, the product topology of the metric spaces X, Y is metrizable. ■

Since we know the order topology on $\mathbb{R} \times \mathbb{R}$ is equal to the product topology on $\mathbb{R}_d \times \mathbb{R}$, and we've already seen in class that the discrete topology and the standard topology are metrizable, the lemma above gives us metrizability of $\mathbb{R} \times \mathbb{R}$ in the dictionary order topology.

■

Question 5

Let X be a metric space with metric d.

(a) Show that $d : X \times X \to \mathbb{R}$ is continuous.
(b) Show that the metric topology on X is the weakest (coarsest, smallest) topology on X relative to which $d : X \times X \to \mathbb{R}$ is continuous.

Answer.

(a) It suffices to show that the d-preimage of basic sets is open. Let $(a, b) \subset \mathbb{R}$ be a basic open set and define

$$W = \{(p, q) \in X \times X : d(p, q) \leq a\}$$
$$U = \{(p, q) \in X \times X : d(p, q) < b\}$$

Now, if $b \leq 0$, then $d^{-1}((a, b)) = \emptyset$ so we're done. Thus, we assume $b > 0$.

It is easy to see that $d^{-1}((a, b)) = W^c \cap U$, for if $(p, q) \in d^{-1}((a, b))$ then $a < d(p, q)$ so $(p, q) \notin W$ and $d(p, q) < b$ so $(p, q) \in U$. Conversely, if $(p, q) \notin W$ and $(p, q) \in U$, then $a < d(p, q) < b$, so $(p, q) \in d^{-1}((a, b))$.

We claim that W is closed, so that W^c is open, and that U is open, so the equation $d^{-1}((a, b)) = W^c \cap U$ writes the d-preimage of (a, b) as an intersection of open sets and hence open.

We first show W closed. Let $(p, q) \in \overline{W}$. It suffices to show that $d(p, q) < a + \varepsilon$ for every $\varepsilon > 0$. Indeed, given $\varepsilon > 0$, the set $B_{\varepsilon/2}(p) \times B_{\varepsilon/2}(q)$ is an open neighborhood of (p, q) and hence must intersect W in some point (x, y), so that

$$\begin{aligned}
d(p, q) &\leq d(p, x) + d(x, q) && \text{triangle inequality} \\
&\leq d(p, x) + d(x, y) + d(y, q) && \text{triangle inequality} \\
&< \frac{\varepsilon}{2} + a + \frac{\varepsilon}{2} && x \in B_{\varepsilon/2}(p), y \in B_{\varepsilon/2}(q), \text{and } (x, y) \in W \\
&< a + \varepsilon
\end{aligned}$$

Therefore, $d(p, q) \leq a$, so $(p, q) \in W$. Since W contains its closure, it is closed.

Next, we show that U is open. We claim the following equality:

$$U = \bigcup_{0<r<b} \bigcup_{x \in X} B_r(x) \times B_{b-r}(x)$$

If (p, q) is in the union, then there exists $0 < r < b$ and $x \in X$ such that $p \in B_r(x)$ and $q \in B_{b-r}(x)$, so that

$$d(p, q) \leq d(p, x) + d(x, q) < r + (b - r) = b$$

and hence $(p, q) \in U$. Conversely, suppose $(p, q) \in U$. Now, if $p = q$, then we can take $r = \frac{b}{2} > 0$ and $x = p = q$ to get (p, q) in the union. Otherwise, $0 < d(p, q) < b$, so setting $r = d(p, q)$ and $x = q$, we get $p \in B_r(q)$ and $q \in B_{b-r}(q)$ and hence (p, q) in the union.

Thus, we can write U as a union of open sets, so U is open.

Since W is closed and U is open, we have $d^{-1}((a, b)) = W^c \cap U$ is a finite intersection of open sets and hence open.

Since the d-preimage of basic sets is open, d is continuous. ∎

(b) Let $\mathcal{T}$ be a topology on X which makes d continuous. We claim that $\mathcal{T}$ contains the basic sets of the metric topology; closure of topologies under finite intersections and arbitrary unions will then show that $\mathcal{T}$ contains the metric topology.

So, let $x \in X$ and $\varepsilon > 0$. Let $(x, \cdot) : X \to X \times X$ be the "pairing" function with x, that is $(x, \cdot)(y) = (x, y)$. Note that this is continuous since composing with the first projection yields a constant function and composing with the second projection yields the identity.

Thus, the composition $d \circ (x, \cdot)$ is also continuous. We claim that

$$B_\varepsilon(x) = (d \circ (x, \cdot))^{-1}(-\infty, \varepsilon)$$

This is easy to see; if $y \in B_\varepsilon(x)$, then

$$(d \circ (x, \cdot))(y) = d((x, \cdot)(y)) = d(x, y) < \varepsilon$$

so $B_\varepsilon(x) \subset (d \circ (x, \cdot))^{-1}(-\infty, \varepsilon)$. Conversely, if $y \in (d \circ (x, \cdot))^{-1}(-\infty, \varepsilon)$, then

$$d(x, y) = (d \circ (x, \cdot))(y) < \varepsilon$$

so $y \in B_\varepsilon(x)$.

Therefore, $B_\varepsilon(x)$ is the preimage of a continuous function and hence open in $\mathcal{T}$. Thus, $\mathcal{T}$ contains every ball, and hence must contain the entirety of the metric topology.

Since any topology making d continuous must contain the metric topology, and the metric topology makes d continuous (by (a)), we have the result: the metric topology on X is the coarsest one making d continuous. ■

Question 6

With the same notation as in Q3, what is the closure of $\mathbb{R}^\infty$ in $\mathbb{R}^\mathbb{N}$ using the uniform topology, defined by the metric $d(x, y) = \sup_k(\min(1, |x_k - y_k|))$

Answer.

Define the set

$$C_0 = \left\{x \in \mathbb{R}^\mathbb{N} \mid \lim_{n\to\infty} x_n = 0\right\}$$

We claim that $\overline{\mathbb{R}^\infty} = C_0$.

($\subset$) Let $x \in \overline{\mathbb{R}^\infty}$. We show x converges to zero. Let $\varepsilon > 0$ be given. Since $x \in \overline{\mathbb{R}^\infty}$, there exists $y \in B_\varepsilon(x) \cap \mathbb{R}^\infty$. Since y is non-zero at only finitely many points, there exists some $N \in \mathbb{N}$ such that $y_n = 0$ if $n > N$. Then, if $n > N$, we have

$$|x_n - 0| = |x_n - y_n| \le d(x, y) < \varepsilon$$

Thus, $\lim_{n\to\infty} x_n = 0$ and so $x \in C_0$.

($\supset$) Let $x \in C_0$. It suffices to show that every ball centered at x intersects $\mathbb{R}^\infty$. So, let $\varepsilon > 0$ and consider the ball $B_\varepsilon(x)$. Since x converges to 0, there exists $N \in \mathbb{N}$ such that $|x_n - 0| < \frac{\varepsilon}{2}$ for $n > N$. Define $y \in \mathbb{R}^\mathbb{N}$ by

$$y_k = \begin{cases} x_k & k \le N \\ 0 & k > N \end{cases}$$

It is clear that $y \in \mathbb{R}^\infty$, for it can be non-zero only in the first N entries. We claim that $y \in B_\varepsilon(x)$. Indeed,

$$\begin{aligned} d(x, y) &= \sup_{k\in\mathbb{N}}(\min(1, |x_k - y_k|)) \\ &= \sup_{k>N}(\min(1, |x_k - y_k|)) \text{ since } x_k = y_k \text{ for } k \le N \\ &= \sup_{k>N}(\min(1, |x_k|)) \qquad y_k = 0 \text{ for } k > N \\ &\le \sup_{k>N}\left(\min\left(1, \frac{\varepsilon}{2}\right)\right) \qquad |x_k| < \frac{\varepsilon}{2} \text{ for } k > N \\ &\le \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Thus, every ball centered at x intersects $\mathbb{R}^\infty$, so every open neighborhood of x intersects $\mathbb{R}^\infty$, and hence $x \in \overline{\mathbb{R}^\infty}$.

Both inclusions show that the closure of $\mathbb{R}^\infty$ in the uniform topology is the set of sequences converging to zero. ∎

Question 7

With the same "uniform metric" as in the previous question, with some fixed $x = (x_1, x_2, ...) \in \mathbb{R}^{\mathbb{N}}$, and with some $0 < r < 1$, let $U(x,r) = \Pi_k(x_k - r, x_k + r)$ (a product of intervals). Show that

(a) $U(x,r)$ is not equal to the ball $B_r(x)$.

(b) $U(x,r)$ is not even open in the uniform topology.

(c) $B_r(x) = \bigcup_{s<r} U(x,s)$.

Answer.

(a) We'll exhibit an element of $U(x,r)$ missing from $B_r(x)$. Consider $y \in \mathbb{R}^{\mathbb{N}}$ defined by

$$y_k = x_k + r\left(1 - \frac{1}{k}\right)$$

Note, then, that $y \in U(x,r)$, since for every $k \in \mathbb{N}$, we have

$$x_k - r < x_k + r\left(1 - \frac{1}{k}\right) < x_k + r$$

On the other hand, $y \notin B_r(x)$, since

$$\begin{aligned} d(x,y) &= \sup_k(\min(1, |x_k - y_k|)) \\ &= \sup_k\left(\min\left(1, r\left(1 - \frac{1}{k}\right)\right)\right) \\ &= \sup_k\left(r\left(1 - \frac{1}{k}\right)\right) && r < 1 \text{ so } r\left(1 - \frac{1}{k}\right) < 1 \\ &= r \end{aligned}$$

So $U(x,r) \neq B(x,r)$. ■

(b) Take $y \in \mathbb{R}^{\mathbb{N}}$ to be as in (a), that is

$$y_k = x_k + r\left(1 - \frac{1}{k}\right)$$

We saw in (a) that $y \in U(x,r)$. If U were open, then there must exist some $\delta > 0$ such that $B_\delta(y) \subset U$. We will show that no such δ can exist.

Let $\delta > 0$ and define $z \in \mathbb{R}^{\mathbb{N}}$ by $z_k = y_k + \delta/2$. Then, $z \in B_\delta(y)$ since

$$\begin{aligned} d(y,z) &= \sup_k(\min(1, |y_k - z_k|)) \\ &= \sup_k(\min(1, \delta/2)) \\ &\leq \frac{\delta}{2} \\ &< \delta \end{aligned}$$

But, $z \notin U(x,r)$, as we now show. Since $r\left(1 - \frac{1}{k}\right) \xrightarrow{k \to \infty} r$, we have some k_0 sufficiently large such that $|r\left(1 - \frac{1}{k_0}\right) - r| < \frac{\delta}{2}$, but then

$$z_{k_0} = y_{k_0} + \frac{\delta}{2} = x_{k_0} + r\left(1 - \frac{1}{k_0}\right) + \frac{\delta}{2} > x_{k_0} + r$$

so that $z_{k_0} \notin U_{k_0}$ and hence $z \notin U(x,r)$.

Since every open ball centered at $y \in U(x,r)$ leaves $U(x,r)$, U can not be open. ■

(c) Let $y \in B_r(x)$. Then, $d(x,y) < r$ so there exists some s such that $d(x,y) < s < r$. Now, for every $k \in \mathbb{N}$, we have

$$|y_k - x_k| \leq d(x,y) < s$$

which implies $y_k \in (x_k - s, x_k + s)$ so that $y \in U(x,s)$ and hence, since $s < r$, $y \in \bigcup_{s<r} U(x,s)$. So, $B_r(x) \subset \bigcup_{s<r} U(x,s)$.

In the other direction, suppose $y \in \bigcup_{s<r} U(x,s)$ so that $y \in U(x,s)$ for some $s < r$. Then, $|y_k - x_k| < s$ for every $k \in \mathbb{N}$, so

$$\begin{aligned} d(x,y) &= \sup_k(\min(1, |x_k - y_k|)) \\ &\leq \sup_k(\min(1, s)) \\ &\leq s \\ &< r \end{aligned}$$

and hence $y \in B_r(x)$. So, $\bigcup_{s<r} U(x,s) \subset B_r(x)$.

Both inclusions show the desired equality: $B_r(x) = \bigcup_{s<r} U(x,s)$. ■

Disclaimers (added October 16th)

- In Q3, when showing $\mathbb{R}^\infty$ is closed, setting

$$U_i = \begin{cases} \mathbb{R} & x_i = 0 \\ \mathbb{R} \setminus \{0\} & \text{otherwise} \end{cases}$$

  makes for a much cleaner proof. Not much changes, but it's easier to argue why $y \in U = \prod_{i \in \mathbb{N}} U_i$ can not be in $\mathbb{R}^\infty$.