MAT327H1 F Homework Assigment 1

October 3, 2024

Question 1.

(a)

Proof. Let $x \in A_0$. We show that $x \in f^{-1}(f(A_0))$. Note that $f(A_0) = \{b | b = f(a) \text{ for at least one } a \in A_0\}$, so it immediately follows that $f(x) \in f(A_0)$. Then, by definition of the preimage of f, $x \in \{a | f(a) \in f(A_0)\} = f^{-1}(f(A_0)$. As x was chosen arbitrarily, $A_0 \subset f^{-1}(f(A_0))$.

Now assume that f is injective. Let $x \in f^{-1}(f(A_0))$. It is enough to show that $x \in A_0$, as we have already shown that $A_0 \subset f^{-1}(f(A_0))$ in the general case. Notice that $x \in \{a | f(a) \in f(A_0)\}$ means that $y = f(x) \in f(A_0)$. Recall the definition of the image of $f(A_0)$, there is some $x_1 \in A_0$ s.t. $y = f(x_1)$. As f is injective, we must have that $x_1 = x$, thus $x \in A_0$ as needed. Since x was chosen arbitrarily, $f^{-1}(f(A_0)) \subset A_0$, and equality follows from the general case (shown above). □

(b)

Proof. Let $y \in f(f^{-1}(B_0))$. We show that $y \in B_0$. By the definition of the image of f, we know there is some $x \in f^{-1}(B_0)$ such that $f(x) = y$. Recall that $f^{-1}(B_0) = \{a | f(a) \in B_0\}$, so it follows that if $x \in f^{-1}(B_0), y = f(x) \in B_0$. Since y was chosen arbitrarily, $f(f^{-1}(B_0)) \subset B_0$.

Now assume that f is surjective. Let $y \in B_0$. It is enough to show that $y \in f(f^{-1}(B_0))$, as we have already shown that $f(f^{-1}(B_0)) \subset B_0$ in the general case. Recall that $f^{-1}(B_0) = \{a | f(a) \in B_0\}$, and since f is surjective there is some $x \in f^{-1}(B_0)$ such that $f(x) = y$. This immediately shows that $y \in f(f^{-1}(B_0))$, because to say $x \in f^{-1}(B_0)$ implies that $y = f(x) \in f(f^{-1}(B_0))$. Since y was chosen arbitrarily, $B_0 \subset f(f^{-1}(B_0))$ and equality follows from the general case (shown above). □

Question 2.

For clarification and justification (if necessary), when I say "the result follows by symmetry", I mean that I have shown a string of if and only if statements, meaning that the exact same proof setup can be used in the other direction. In the context of this problem set, it means fixing a point in the other set, and "going the other way" will follow from what was already shown. I did it to save space and time; I hope this is acceptable.

(a)

Proof. Assume $B_0 \subset B_1$. We show that $f^{-1}(B_0) \subset f^{-1}(B_1)$. Let $x \in f^{-1}(B_0)$, then $f(x) \in B_0$ by definition of the preimage of f. Then since $B_0 \subset B_1$, $f(x) \in B_1$ meaning that $x \in f^{-1}(B_1)$, also by definition of the preimage of f. Since x was chosen arbitrarily, $f^{-1}(B_0) \subset f^{-1}(B_1)$ as needed. □

(b)

Proof. First we show that $f^{-1}(B_0 \cup B_1) \subset f^{-1}(B_0) \cup f^{-1}(B_1)$. Let $x \in f^{-1}(B_0 \cup B_1)$, then $f(x) \in B_0 \cup B_1$, or equivalently we can say $f(x) \in B_0$ or $f(x) \in B_1$, by definition of the preimage of f. This is equivalent to saying that, $x \in f^{-1}(B_0)$ or $x \in f^{-1}(B_1)$ by definition of the image of f, which is also equivalent to saying $x \in f^{-1}(B_0) \cup f^{-1}(B_1)$. Since x was chosen arbitrarily, $f^{-1}(B_0 \cup B_1) \subset f^{-1}(B_0) \cup f^{-1}(B_1)$. The fact that $f^{-1}(B_0 \cup B_1) \supset f^{-1}(B_0) \cup f^{-1}(B_1)$ (and thus the result) follows by symmetry.

□

(c)

Proof. First we show that $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$. Let $x \in f^{-1}(B_0 \cap B_1)$. Then equivalenty, $f(x) \in B_0 \cap B_1$, by definition of the preimage of f. Equivalently we can say $f(x) \in B_0$ and $f(x) \in B_1$ and by definition of the image of f, this is equivalent to the statement ($x \in f^{-1}(B_0)$ and $x \in f^{-1}(B_1)$), which is then equivalent to saying $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$. Since x was chosen arbitrarily, $f^{-1}(B_0 \cap B_1) \subset f^{-1}(B_0) \cap f^{-1}(B_1)$. The fact that $f^{-1}(B_0 \cap B_1) \supset f^{-1}(B_0) \cap f^{-1}(B_1)$ (and thus the result) follows by symmetry. □

(d)

Proof. First we show that $f^{-1}(B_0 \setminus B_1) \subset f^{-1}(B_0) \setminus f^{-1}(B_1)$. Let $x \in f^{-1}(B_0 \setminus B_1)$. Then by definition of the preimage of f, this is equivalent to saying that $f(x) \in B_0 \setminus B_1$, or (also equivalently) we can say $f(x) \in B_0$ and $f(x) \notin B_1$. By definition of the image of f, this is then equivalent to saying that $x \in f^{-1}(B_0)$ and $x \notin f^{-1}(B_1)$ which is also then equivalent to saying $x \in f^{-1}(B_0) \setminus f^{-1}(B_1)$. Since x was chosen arbitrarily, $f^{-1}(B_0 \setminus B_1) \subset f^{-1}(B_0) \setminus f^{-1}(B_1)$. The fact that $f^{-1}(B_0 \setminus B_1) \supset f^{-1}(B_0) \setminus f^{-1}(B_1)$ (and thus the result) follows by symmetry. □

(e)

Proof. Assume that $A_0 \subset A_1$. Let $y \in f(A_0)$ so that by definition of the image of f, $\exists x \in A_0$ s.t. $y = f(x)$. But since $A_0 \subset A_1, x \in A_1$, therefore $y = f(x) \in f(A_1)$, by definition of the image of f. Since y was chosen arbitrarily, indeed $f(A_0) \subset f(A_1)$. □

(f)

Proof. First we show that $f(A_0 \cup A_1) \subset f(A_0) \cup f(A_1)$. Let $y \in f(A_0 \cup A_1)$, then by the definition of the image of f, this is equivalent to the statement, $(\exists x \in A_0 \cup A_1$ s.t. $y = f(x)) \iff (\exists x \in A_0$ or $x \in A_1$ s.t. $y = f(x)) \iff (f(x) \in f(A_0)$ or $f(x) \in f(A_1))$. The last biconditional follows from the definition of the image of f, and is equivalent to the statement $f(x) \in f(A_0) \cup f(A_1)$. The fact that $f(A_0 \cup A_1) \supset f(A_0) \cup f(A_1)$ (and thus the result) follows by symmetry. □

(g)

Proof. We show that $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$. Let $y \in f(A_0 \cap A_1)$, then by definition of the image of f, $\exists x \in A_0 \cap A_1$ s.t. $y = f(x)$. This means that $x \in A_0$ and $x \in A_1$. Then by the definition of the image of f, it must be the case that both $y = f(x) \in f(A_0)$ and $y = f(x) \in f(A_1)$, which is equivalent to the statement $y = f(x) \in f(A_0) \cap f(A_1)$. Since y was chosen arbitrarily, $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$.

Now suppose that f is injective. We will show that $f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1)$ so that equality is implied by the general case. Let $y \in f(A_0) \cap f(A_1)$, then by definition of the image of f, $\exists x \in A_0 \cap A_1$ s.t. $f(x) = y$. We must have $x \in A_0 \cap A_1$ because if $x \in A_0 \setminus A_1$, then by definition of the image of f, $\exists x' \in A_1$ s.t. $f(x') = y$, which contradicts the injectivity of f (the same holds when assuming $x \in A_1 \setminus A_0$). So since $x \in A_0 \cap A_1$, we must have, by definition of the image of f, that $y = f(x) \in f(A_0 \cap A_1)$. Since y was chosen arbitrarily, $f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1)$. Paired with the general case, we have shown that $f(A_0) \cap f(A_1) = f(A_0 \cap A_1)$ when f is injective. □

(h)

Proof. We show that $f(A_0) \setminus f(A_1) \subset f(A_0 \setminus A_1)$. Let $y \in f(A_0) \setminus f(A_1)$, then by definition of the image of f, $\exists x \in A_0$ s.t. $y = f(x)$. Additionally, we assert that $\nexists x_1 \in A_1$ s.t. $y = f(x)$, as this would imply that $y = f(x) \in f(A_1)$, a contradiction. So it must be the case that $x \in A_0 \setminus A_1$, meaning that $y = f(x) \in f(A_0 \setminus A_1)$. Since y was chosen arbitrarily, $f(A_0) \setminus f(A_1) \subset f(A_0 \setminus A_1)$.

Now suppose that f is injective. We will show that $f(A_0 \setminus A_1) \subset f(A_0) \setminus f(A_1)$ so that equality is implied by the general case. Let $y \in f(A_0 \setminus A_1)$, then by definition of the image of f, $\exists x \in A_0 \setminus A_1$ s.t. $f(x) = y$. It then follows, by definition of the image of f, that $f(x) \in f(A_0)$, and we must show why $f(x) \notin f(A_1)$. To see why this is so, notice that if $\exists x_1 \in A_1$ s.t. $y = f(x)$, then by the injectivity of f, $x = x_1$, implying that $x \in A_1$, a contradiction. We conclude, again by definition of the image of f, that $y = f(x) \notin f(A_1)$, meaning that $y = f(x) \in f(A_0) \setminus f(A_1)$. Since y was chosen arbitrarily, $f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1)$. Paired with the general case, we have shown that $f(A_0) \cap f(A_1) = f(A_0 \cap A_1)$ when f is injective. □

Question 3.

For clarification and justification (if necessary), when I say "the result follows by symmetry", I mean that I have shown a string of if and only if statements, meaning that the exact same proof setup can be used in the other direction. In the context of this problem set, it means fixing a point in the other set, and "going the other way" will follow from what was already shown. I did it to save space and time; I hope this is acceptable.

(b)

Proof. Let $\{B_\beta\}_{\beta\in J}$ be a family of subsets of B, indexed by J. We will start by showing that $f^{-1}(\bigcup_\beta B_\beta) \subset \bigcup_\beta(f^{-1}(B_\beta))$. Let $x \in f^{-1}(\bigcup_\beta B_\beta)$, which is equivalent to the statement, $f(x) \in \bigcup_\beta B_\beta$, by definition of the preimage of f. Notice that $(f(x) \in \bigcup_\beta B_\beta) \iff (\exists\beta \in J \text{ s.t. } f(x) \in B_\beta) \iff (\exists\beta \in J \text{ s.t. } x \in f^{-1}(B_\beta))$. The last biconditional follows from the definition of the preimage of f, and is then equivalent to the statement $x \in \bigcup_\beta(f^{-1}B_\beta)$. Since x was chosen arbitrarily, $f^{-1}(\bigcup_\beta B_\beta) \subset \bigcup_\beta(f^{-1}(B_\beta))$. The fact that $f(\bigcup_\beta B_\beta) \supset \bigcup_\beta(f^{-1}(B_\beta))$ (and thus the result) follows by symmetry. □

(c)

Proof. As before, Let $\{B_\beta\}_{\beta\in J}$ be a family of subsets of B, indexed by J. We will start by showing that $f^{-1}(\bigcap_\beta B_\beta) \subset \bigcap_\beta(f^{-1}(B_\beta))$. Let $x \in f^{-1}(\bigcap_\beta B_\beta)$, which is equivalent to the statement, $f(x) \in \bigcap_\beta B_\beta$, by definition of the preimage of f. Notice that $(f(x) \in \bigcap_\beta B_\beta) \iff (\forall\beta \in J,\ f(x) \in B_\beta) \iff (\forall\beta \in J,\ x \in f^{-1}(B_\beta))$. The last biconditional follows from the definition of the preimage of f, and is then equivalent to the statement $x \in \bigcap_\beta(f^{-1}B_\beta)$. Since x was chosen arbitrarily, $f^{-1}(\bigcap_\beta B_\beta) \subset \bigcap_\beta(f^{-1}(B_\beta))$. The fact that $f(\bigcap_\beta B_\beta) \supset \bigcap_\beta(f^{-1}(B_\beta))$ (and thus the result) follows by symmetry. □

(f)

Proof. Let $\{A_\alpha\}_{\alpha\in J}$ be a family of subsets of A, indexed by J. We will start by showing that $f(\bigcup_\alpha A_\alpha) \subset \bigcup_\alpha f(A_\alpha)$. Let $y \in f(\bigcup_\alpha A_\alpha)$, which is equivalent to the statement $\exists x \in \bigcup_\alpha A_\alpha$ s.t. $y = f(x)$, by definition of the image of f. Then notice that $(\exists x \in \bigcup_\alpha A_\alpha \text{ s.t. } y = f(x)) \iff (\exists x \in A_\alpha \text{ s.t. } y = f(x) \text{ for some } \alpha \in J) \iff (\exists\alpha \in J \text{ s.t. } y = f(x) \in f(A_\alpha))$. The last biconditional follows from the definition of the image of f, and is equivalent to the statement $y = f(x) \in \bigcup_\alpha f(A_\alpha)$. Since y was chosen arbitrarily, $f(\bigcup_\alpha A_\alpha) \subset \bigcup_\alpha f(A_\alpha)$. The fact that $f(\bigcup_\alpha A_\alpha) \supset \bigcup_\alpha f(A_\alpha)$ (and thus the result) follows by symmetry. □

(h)

Proof. As before, let $\{A_\alpha\}_{\alpha\in J}$ be a family of subsets of A, indexed by J. We will start by showing that $f(\bigcap A_\alpha) \subset \bigcap_\alpha f(A_\alpha)$. Let $y \in f(\bigcap A_\alpha)$, then, using the definition of the image of f, $\exists x \in \bigcap_\alpha A_\alpha$ s.t. $y = f(x)$ so $x \in A_\alpha\ \forall\alpha \in J$. This then implies that $y = f(x) \in f(\bigcap_\alpha A_\alpha)$, also by definition of the image. Since y was chosen arbitrarily, $f(\bigcap A_\alpha) \subset \bigcap_\alpha f(A_\alpha)$.

Now suppose that f is injective. We will show that $\bigcap_\alpha f(A_\alpha) \subset f(\bigcap A_\alpha)$ so that equality is implied by the general case (shown above). Let $y \in \bigcap_\alpha f(A_\alpha)$, so $y \in f(A_\alpha)\ \forall\alpha \in J$. So, for some particular $\alpha_0 \in J$, $\exists x_0 \in A_{\alpha_0}$ s.t. $f(x_0) = y$. Since f is injective, this x_0 must be unique and therefore we must have $x_0 \in \bigcap_\alpha A_\alpha$. To see this, suppose for the sake of contradiction that for some $\alpha_i \in J$, $x_0 \notin A_{\alpha_i}$. Since $y \in f(A_{\alpha_i})$, there must be some $x_i \in A_{\alpha_i}$ s.t. $f(x_i) = y$ by the definition of the image of f. Injectivity implies that $x_i = x_0$, meaning $x_0 \in A_{\alpha_i}$, a contradiction. So $x_0 \in \bigcap_\alpha A_\alpha$, which then implies that $y = f(x_0) \in f(\bigcap_\alpha A_\alpha)$ by definition of the image of f. Since y was chosen arbitrarily, this shows that $\bigcap_\alpha f(A_\alpha) \subset f(\bigcap A_\alpha)$. Paired with the general case, we have thus shown that $\bigcap_\alpha f(A_\alpha) = f(\bigcap A_\alpha)$ when f is injective. $\square$