This is a preview of what students will see when they are submitting the assignment. Interactive features are disabled.

Homework Assignment 8

Solve and submit your solutions of the following problems. Note that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Due date
Tuesday, November 19, 2024 11:59 pm (Eastern Standard Time)

Late penalty
5% deducted per hour

Q1 (0 points)

Read sections 51-55 in Munkres' textbook (Topology, 2nd edition). Remember that reading math isn't like reading a novel! If you read a novel and miss a few details most likely you'll still understand the novel. But if you miss a few details in a math text, often you'll miss everything that follows. So reading math takes reading and rereading and rerereading and a lot of thought about what you've read.

Q2 (30 points)

If X and Y are topological spaces, we say that two continuous functions $F_0 : X \to Y$ and $F_1 : X \to Y$ are *homotopic* if there exists a continuous function $H : X \times I \to Y$ such that for all $x \in X$, $H(x, 0) = F_0(x)$ and $H(x, 1) = F_1(x)$. In that case, we write $F_0 \sim F_1$.

(a) Prove that the relation $\sim$ is an equivalence relation.

(b) If γ is a path in X and if $F : X \to Y$ is continuous, we define a path in Y by $F_*\gamma := F \circ \gamma$. Show that if γ_0 and γ_1 are path homotopic in X then $F_*\gamma_0$ and $F_*\gamma_1$ are path homotopic in Y.

(c) Prove that if $F_i : X \to Y$ and $G_i : Y \to Z$ are continuous for $i = 0, 1$, and if $F_0 \sim F_1$ and $G_0 \sim G_1$, then $G_0 \circ F_0 \sim G_1 \circ F_1$.

Q3 (20 points)

Let X be a path connected space, and recall that we say that X is simply-connected if for some $x_0 \in X$, the group $\pi_1(X, x_0)$ is trivial (and in that case, for any $x_0 \in X$ the group $\pi_1(X, x_0)$ is trivial). Recall also that $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ is the unit circle in $\mathbb{R}^2$.

(a) Show that X is simply connected if and only if any two paths γ_0 and γ_1 in X that have the same endpoints (namely, $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$) are path homotopic.

(b) Show that X is simply connected if and only if, using the same language as in the previous question, every continuous function $\lambda : S^1 \to X$ is homotopic to a constant function. **Warning.** A homotopy from such a λ to a constant function need not keep any basepoint in place.

Q4 (30 points)

Recall that $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ is the unit sphere in $\mathbb{R}^3$. It is well known (and can be used freely) that S^2 with one point removed is homeomorphic to $\mathbb{R}^2$.

(a) Show that any continuous $\lambda_0 : S^1 \to S^2$ which is not surjective is homotopic to a constant function. (**Warning.** There actually exist continuous surjective maps $\lambda_0 : S^1 \to S^2$! If you don't believe, web-search "Peano Curve" and complete the missing details on your own).

(b) Show that any continuous $\lambda : S^1 \to S^2$ is homotopic to a continuous $\lambda_0 : S^1 \to S^2$ which is not surjective.

(c) With the same language as in the previous exercise, deduce that S^2 is simply connected.