

Khovanov: $K(L)$ is a chain complex of graded \mathbb{Z} -modules;

$$V = \text{span}\langle v_+, v_- \rangle; \quad \text{deg } v_{\pm} = \pm 1; \quad \text{qdim } V = q + q^{-1};$$

$$K(\bigcirc^k) = V^{\otimes k}; \quad K(\text{↗}) = \text{Flatten} \left(0 \rightarrow K(\bigcirc)\{1\} \rightarrow K(\text{↘})\{2\} \rightarrow 0 \right);$$

height 0 height 1

$$K(\text{↖}) = \text{Flatten} \left(0 \rightarrow K(\text{↘})\{-2\} \rightarrow K(\bigcirc)\{-1\} \rightarrow 0 \right);$$

height -1 height 0

$$\left(\bigcirc \bigcirc \xrightarrow{\text{cup}} \text{cup} \right) \rightarrow (V \otimes V \xrightarrow{m} V)$$

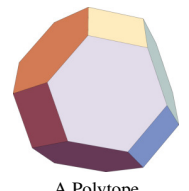
$$m : \begin{cases} v_+ \otimes v_- \mapsto v_- & v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_+ \mapsto v_- & v_- \otimes v_- \mapsto 0 \end{cases}$$

$$\left(\text{cup} \xrightarrow{\text{cap}} \bigcirc \bigcirc \right) \rightarrow (V \xrightarrow{\Delta} V \otimes V)$$

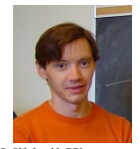
$$\Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases}$$

$$\bigcirc^k \mapsto (q + q^{-1})^k$$

$$J : \text{↗} \mapsto q)(-q^2 \text{↘}, \quad J : \text{↖} \mapsto -q^{-2} \text{↘} + q^{-1} \text{↘}$$



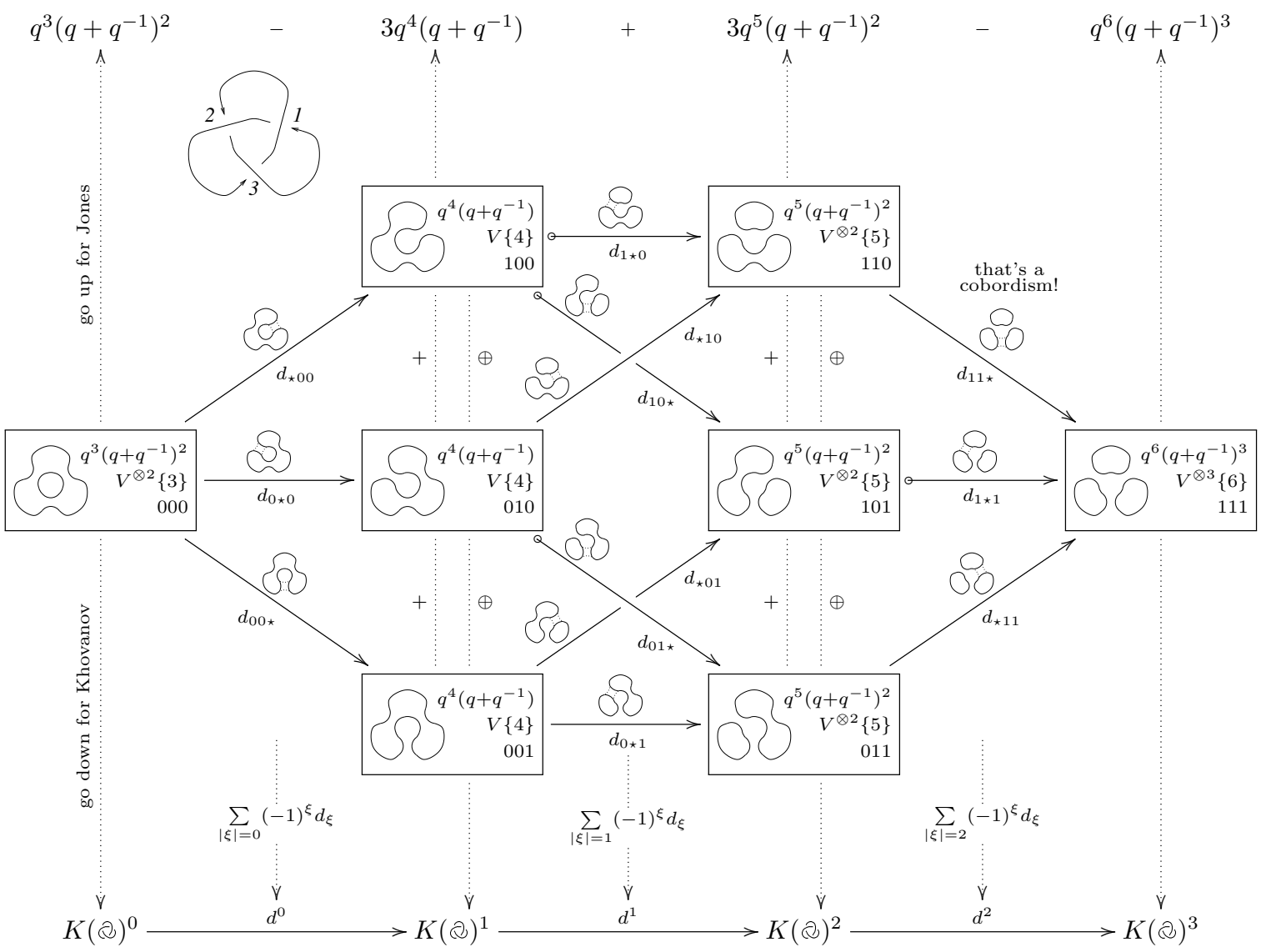
A Polytope



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Example:

$$= q + q^3 + q^5 - q^9.$$



(here $(-1)^\xi := (-1)^{\sum_{i < j} \xi_i}$ if $\xi_j = \star$) = $K(\text{⊙})$.

Theorem 1. The graded Euler characteristic of $K(L)$ is $J(L)$.

Theorem 2. The homology $\text{Kh}(L)$ of $K(L)$ is a link invariant.

Theorem 3. $\text{Kh}(L)$ is strictly stronger than $J(L)$: $J(\bar{5}_1) = J(10_{132})$ yet $\text{Kh}(\bar{5}_1) \neq \text{Kh}(10_{132})$.

Theorem 4. Kh extends to a functor defined on the category of 2D cobordisms in \mathbb{R}^4 between links in \mathbb{R}^3 .

References. Khovanov's [arXiv:math.QA/9908171](https://arxiv.org/abs/math.QA/9908171) and [arXiv:math.QA/0103190](https://arxiv.org/abs/math.QA/0103190) and my <http://www.math.toronto.edu/~drorbn/papers/Categorification/>.