Dror Bar-Natan: Classes: 2021-22: MAT 257 Analysis II:

A Bit on Maxwell's Equations

There's also a handout at http://drorbn.net/2122-257/ap/Maxwell.pdf

Better w/ video on !

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A Bit on Maxwell's Equations

Prerequisites.

- Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.
- Integration by parts: $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg \omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$ for compactly supported forms.
- The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.
- The simplesest least action principle: the extremes of $q \mapsto S(q) = \int_a^b \left(\frac{1}{2} m\dot{q}^2(t) - V(q(t))\right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

Table 18–1 Classical Physics

Maxwell's equations

I. $\nabla \cdot E = \frac{\rho}{\epsilon_0}$ (Flux of E through a closed surface) = (Charge inside)$/\epsilon_0$

II. $\nabla \times E = -\frac{\partial B}{\partial t}$ (Line integral of E around a loop) $= -\frac{d}{dt}$(Flux of B through the loop)

III. $\nabla \cdot B = 0$ (Flux of B through a closed surface) = 0

IV. $c^2 \nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}$ c^2(Integral of B around a loop) = (Current through the loop)$/\epsilon_0$ $+\frac{\partial}{\partial t}$(Flux of E through the loop)

[Conservation of charge
$\nabla \cdot j = -\frac{\partial \rho}{\partial t}$ (Flux of current through a closed surface) $= -\frac{\partial}{\partial t}$(Charge inside)]

Force law
$F = q(E + v \times B)$

Law of motion
$\frac{d}{dt}(p) = F,$ where $p = \frac{mv}{\sqrt{1 - v^2/c^2}}$ (Newton's law, with Einstein's modification)

Gravitation
$F = -G\frac{m_1 m_2}{r^2} e_r$

The Feynman Lectures on Physics vol. II, page 18-2

The Action Principle. The *4-Vector Potential* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} \|dA\|^2\, dtdxdydz + J \wedge A}$$

where the 3-form J is the *charge-current*.

The Euler-Lagrange Equations in this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

These are the Maxwell equations! Indeed, writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$dJ = 0 \Longrightarrow$	$\operatorname{div} j = -\frac{\partial \rho}{\partial t}$	"conservation of charge"
$dF = 0 \Longrightarrow$	$\operatorname{div} B = 0$	"no magnetic monopoles"
	$\operatorname{curl} E = -\frac{\partial B}{\partial t}$	that's how generators work!
$d * F = J \Longrightarrow$	$\operatorname{div} E = -\rho$	"electrostatics"
	$\operatorname{curl} B = j - \frac{\partial E}{\partial t}$	that's how electromagnets work!

Exercise. Use the Lorentz metric to fix the sign errors.

Exercise. Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers.

Exercise. With $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ use $S = mc\int_{e_1}^{e_2}(ds + eA)$ to derive Feynman's "law of motion" and "force law".

Table 18–1 Classical Physics

Maxwell's equations

I.	$\nabla \cdot \boldsymbol{E} = \dfrac{\rho}{\epsilon_0}$	(Flux of $\boldsymbol{E}$ through a closed surface) $=$ (Charge inside)$/\epsilon_0$
II.	$\nabla \times \boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$	(Line integral of $\boldsymbol{E}$ around a loop) $= -\dfrac{d}{dt}$ (Flux of $\boldsymbol{B}$ through the loop)
III.	$\nabla \cdot \boldsymbol{B} = 0$	(Flux of $\boldsymbol{B}$ through a closed surface) $= 0$
IV.	$c^2\nabla \times \boldsymbol{B} = \dfrac{\boldsymbol{j}}{\epsilon_0} + \dfrac{\partial \boldsymbol{E}}{\partial t}$	c^2(Integral of $\boldsymbol{B}$ around a loop) $=$ (Current through the loop)$/\epsilon_0$ $+\dfrac{\partial}{\partial t}$ (Flux of $\boldsymbol{E}$ through the loop)

Conservation of charge

$$\nabla \cdot \boldsymbol{j} = -\frac{\partial \rho}{\partial t}$$

(Flux of current through a closed surface) $= -\dfrac{\partial}{\partial t}$ (Charge inside)

Force law

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Law of motion

$$\frac{d}{dt}(\boldsymbol{p}) = \boldsymbol{F}, \quad \text{where} \quad \boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}$$

(Newton's law, with Einstein's modification)

Gravitation

$$\boldsymbol{F} = -G\,\frac{m_1 m_2}{r^2}\,\boldsymbol{e}_r$$

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Prerequisites.

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- The simplesest least action principle: the extremes of $q \mapsto S(q) = \int_a^b \left(\frac{1}{2} m\dot{q}^2(t) - V(q(t)) \right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

Prerequisite 1.

$$\begin{tikzcd} \Lambda^{k-1} \arrow[r, "d"] \arrow[rr, bend right, "0"'] & \Lambda^{k} \arrow[r, "d"] & \Lambda^{k+1} \end{tikzcd}$$

Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.

$\operatorname{im} d \subset \ker d$

$\operatorname{im} d = \ker d$ on $\mathbb{R}^n$

$\mathbb{R}^2 \setminus \{0\}$

Prerequisite 2.

$$\int_a^b f g' = \cancel{fg\Big|_a^b} - \int f' g$$

Integration by parts: $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg \omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$ for compactly supported forms.

$$\int_{B(r)} - = \int_{\mathbb{R}^n} d(\omega \wedge \eta) = \int_{\mathbb{R}^n} d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$$

$$\| \qquad \int_{\partial B(r)} \omega \wedge \eta = 0$$

Prerequisite 3.

$\dim \Lambda^k(\mathbb{R}^n) = \binom{n}{k} = \binom{n}{n-k}$

1
1 2 1
1 3 3 1
1 4 6 4 1

$\Lambda^k \sim \Lambda^{n-k}$?

$\{dx_I\}_{I \subset n, |I| = k}$ $\quad dx_I \rightsquigarrow dx_{I^c}$

The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.

$* : \Lambda^k \longrightarrow \Lambda^{n-k}$

$w \wedge * w = |w|^2 dx_1 \ldots dx_n$

On $\mathbb{R}^4_{txyz}$

$*(dt \wedge dx) = + dy \wedge dz$

$\langle v_i, v_j \rangle = \ldots$

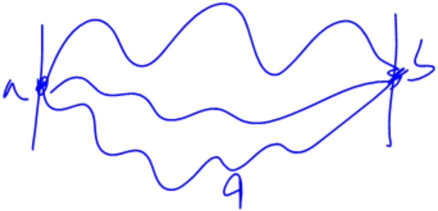
$\langle e_i, e_j \rangle = \delta_{ij}$

$\rightsquigarrow \{\varphi_i\}$ dual to $\{e_j\}$

dual $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$

$\langle \varphi_I, \varphi_J \rangle = \delta_{IJ}$

Prerequisite 4.



The simplesest ~~least~~ extreme action principle: the extremes of $q \mapsto S(q) = \int_a^b \left(\underline{\tfrac{1}{2} m\dot{q}^2(t)} - \underline{V(q(t))} \right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

The Action Principle.

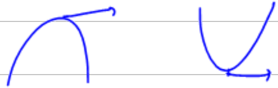
The *4-Vector Potential* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} |dA|^2 \, dtdxdydz + J \wedge A}$$

where the 3-form J is the *charge-current*.

$f : \mathbb{R}^n \to \mathbb{R}$ $\quad a \in \mathbb{R}^n$ is critical.

if $f'(a) = 0 \iff$



$$f(a+\epsilon b) = f(a) + \epsilon \underbrace{\cdots}_{=\,0} + o(\epsilon^2)$$

A is critical for $S_J(A) = \int_{\mathbb{R}^4} \frac{1}{2}|dA|^2\, dt\,dx\,dy\,dz + J \wedge A$

$\iff \forall B$ $S_J(A+\epsilon B)$ has no term prop to ϵ.

$$S_J(A+\epsilon B) = \int_{\mathbb{R}^4} \frac{1}{2}\langle dA+\epsilon dB, dA+\epsilon dB\rangle + J\wedge(A+\epsilon B)$$

$$0 = \begin{matrix}\text{Part prop}\\ \text{to } \epsilon\end{matrix} = \frac{\cancel{1}\,2}{2}\int_{\mathbb{R}^4} \langle dA, dB\rangle + J\wedge B$$

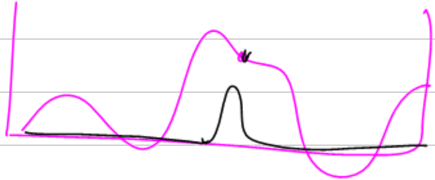
$$= \int_{\mathbb{R}^4} \langle dB, dA\rangle dt\,dx\,dy\,dz + J\wedge B$$

$$= \int_{\mathbb{R}^4} dB\wedge * dA + \underbrace{J\wedge B}_{-B\wedge J}$$

$$= + \int_{\mathbb{R}^4} B \wedge \underbrace{(d * dA - J)} = 0 \quad \forall B$$

$$\Rightarrow d * dA - J = 0$$

$$\int f \cdot g = 0 \quad \forall g \Rightarrow f = 0$$



$$\boxed{d\underbrace{*dA}_{\substack{\text{1-form}\\ \text{2-form}}} = J} \iff \text{Feynman's eqns 1–5}$$

$$\Lambda^4 \ni \boxed{dJ = 0}$$ cond. for having solutions

Write $dA = F \in \Lambda^2(\mathbb{R}^4)$

$$\boxed{d*F = J} \in \Lambda^3$$

$$\boxed{dF = 0} \in \Lambda^3$$

The Euler-Lagrange Equations

in this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

These are the Maxwell equations!

$F \mapsto (\vec{E}, \vec{B})$ magnetic field

Electric field

$J \leftrightarrow (\rho, j)$

charge current

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

Writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

charge

$dJ = 0 \implies$	$\operatorname{div} j = -\dfrac{\partial \rho}{\partial t}$	"conservation of charge"
$dF = 0 \implies$	$\operatorname{div} B = 0$	"no magnetic monopoles"
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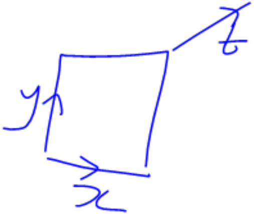
$dJ = 0$



$$\boxed{dJ = 0 \qquad dF = 0 \qquad d\star F = J}$$

with $J = \rho\, dxdydz - j_x\, dydzdt - j_y\, dzdxdt - j_z\, dxdydt$.

$$dJ = 0 \implies \boxed{\operatorname{div} j = -\frac{\partial \rho}{\partial t}} \qquad \text{“conservation of charge”}$$

$$dJ = \left(\frac{\partial \rho}{\partial t} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}\right) dt\,dx\,dy\,dz$$

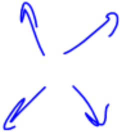
$$\frac{\partial \rho}{\partial t} + \operatorname{div} j = 0$$

$$dF = 0 = (\partial_x B_x + \partial_y B_y + \partial_z B_z)\,dx\,dy\,dz + (\partial_z E_y - \partial_y E_z + \partial_t B_x)\,dt\,dy\,dz + (\quad) + (\quad)$$

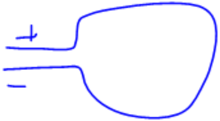
Ω^3

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

with $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and
$J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$.



$dF = 0 \Longrightarrow$	$\operatorname{div} B = 0$	"no magnetic monopoles"
	$\operatorname{curl} E = -\dfrac{\partial B}{\partial t}$	that's how generators work!



$d * F = J$

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

with $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and
$J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$.
With $\omega \wedge *\omega = |\omega|^2 dtdxdydz$ we have

$dxdt(dydz)$ $$*dxdt = -dydz, \qquad *dydt = -dzdx, \qquad *dzdt = -dxdy,$$

$$*dydz = -dxdt, \qquad *dzdx = -dydt, \qquad *dxdy = -dxdt,$$

so $*F = (-B_x dxdt - B_y dydt - B_z dzdt) + (-E_x dydz - E_y dzdx - E_z dxdy)$.

$d * F = J \Longrightarrow$	$\operatorname{div} E = -\rho$	"electrostatics"
	$\operatorname{curl} B = j - \dfrac{\partial E}{\partial t}$	that's how electromagnets work!

Table 18–1 Classical Physics

Maxwell's equations

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III.	$\nabla \cdot B = 0$	(Flux of B through a closed surface) = 0
IV.	$c^2 \nabla \times B = \dfrac{j}{\epsilon_0} + \dfrac{\partial E}{\partial t}$	c^2(Integral of B around a loop) = (Current through the loop)/ϵ_0 $+ \dfrac{\partial}{\partial t}$(Flux of E through the loop)

$dF = 0$

$d{*}F = J$

Conservation of charge

$\nabla \cdot j = -\dfrac{\partial \rho}{\partial t}$ (Flux of current through a closed surface) = $-\dfrac{\partial}{\partial t}$ (Charge inside)

$dJ = 0$

Force law

$F = q(E + v \times B)$

Law of motion

$\dfrac{d}{dt}(p) = F,$ where $p = \dfrac{mv}{\sqrt{1 - v^2/c^2}}$ (Newton's law, with Einstein's modification)

Gravitation

$F = -G \dfrac{m_1 m_2}{r^2} e_r$

Feynman again. But wait, in our last two equations the sign of E is wrong!

Exercise 1.

Euclidean metric $x^2+y^2+z^2+t^2$

Lorentz metric $x^2+y^2+z^2-t^2$

for physicists.

$*^E \longrightarrow *^L$

Use the Lorentz metric to fix the sign errors.

Exercise 2.

Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers.

$$L : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

$$A_1 = L^* A_2$$

$$F_1 = L^* F_2$$

Exercise 3.

With $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ use $S = mc\int_{e_1}^{e_2}(ds + eA)$ to derive Feynman's "law of motion" and "force law".