# Final Exam Information and Practice Questions

- The Final Exam will take place on Wednesday April 27, 7-10pm in person at EX200.
- Our TAs Sebastian and Shuyang will hold extra pre-exam office hours, in their usual zoom rooms. Sebastian on Wednesday April 20 and Friday April 22 at 11am-1pm at Sebastian's Zoom (password vchat), and Shuyang on Monday, Tuesday, and Wednesday April 25-27 at 11am-1pm at Shuyang's Zoom (password vchat).
- I will hold my regular office hours on Tuesday April 12, at 9:30-10:30am, I will skip my regular office hours on April 19 and 26, and I will hold extra office hours on Monday, Tuesday, and Wednesday April 25-27 1:30-4pm. All these in person at BA 6178 and online at `http://drorbn.net/vchat`.
- Material: *Everything* excluding the material on Maxwell equations, with very light further emphasis on the material not covered in the previous term tests. Very roughly, one third of the questions will come straight from class material, one third from HW, and the last third will be fresh questions.
- No outside material will be allowed other than stationery, minimal hydration and snacks, and stuffed animals.
- The format will be "Solve 7 of 7", or maybe "6 of 6" or "8 of 8", in 3 hours.
- To prepare: Consider last year's final assessment and last year's "reject" questions, yet bear in mind that these are all "fresh" questions as last year the test was online. Much more important: make sure that you understand every single bit of class material!

The 2020/21 Final Assessment:

1. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable at a point $a \in \mathbb{R}^n$, and for $b \in \mathbb{R}^n$ define $L(b) := \lim_{\epsilon\to 0} \frac{f(a+\epsilon b)-f(a)}{\epsilon}$. Prove that $L$ is a linear function of $b$.
2. A function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies $|(x-y)-(f(x)-f(y))| \leq \frac{1}{3}|x-y|$ for every $x, y \in \mathbb{R}^n$. Prove that $f$ is continuous.
3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function whose support is contained in the unit square $[0,1]\times[0,1]$ in $\mathbb{R}^2$, let $s\colon [0,1] \to [0,1]$ be a continuous function, and define $g\colon \mathbb{R}^2 \to \mathbb{R}$ by $g(x,y) = f(x, y-s(x))$. Explain why $f$ is integrable on $[0,1]\times[0,1]$ and why $g$ is integrable on $[0,1]\times[0,2]$, and show that
$$\int_{[0,1]\times[0,1]} f = \int_{[0,1]\times[0,2]} g.$$
*Hint.* Make Fubini happy!
4. If $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ is a smooth path and $t \in \mathbb{R}$, let $\dot\gamma(t)$ be the tangent vector to $\mathbb{R}^n$ given as the pair $(\gamma(t), \gamma'(t)e_1)$, where $e_1$ is the standard basis vector of $\mathbb{R}$. Show
(a) If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then $D_{\dot\gamma(t)}f = (f\circ\gamma)'(t)$, where $D$ denotes the directional derivative.
(b) If $g\colon \mathbb{R}^n \to \mathbb{R}^m$ is smooth and $\eta = g\circ\gamma$, then $\dot\eta(t) = g_*(\dot\gamma(t))$.
5. Let $v_1 = \begin{pmatrix}3\\1\end{pmatrix}$ and $v_2 = \begin{pmatrix}5\\2\end{pmatrix}$. Together, they form a basis $(v_1, v_2)$ of $\mathbb{R}^2$. Write the dual basis $(\varphi_1, \varphi_2)$ as a pair of row vectors.

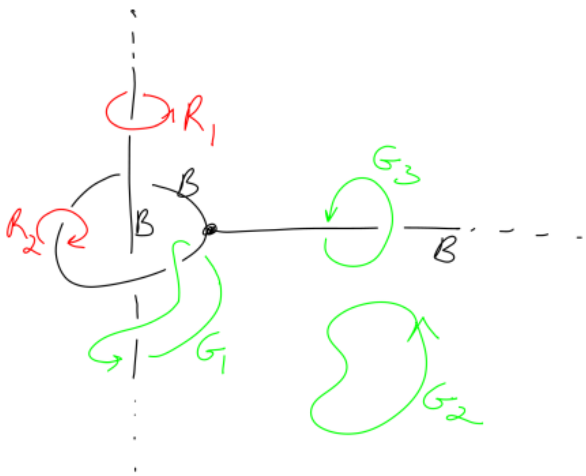
6. Let $c_1$ and $c_2$ be singular 1-cubes in $\mathbb{R}^7$, for which $c_1(0) = c_1(1)$ and $c_2(0) = c_2(1)$.

   (a) Show that there is a singular 2-cube $c$ in $\mathbb{R}^7$ for which $\partial c = c_1 - c_2$.

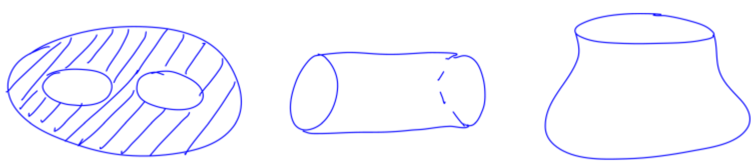
   (b) Suppose now that $c_1(0) = c_1(1)$ but $c_2(0) \neq c_2(1)$. Is it still possible that there is a singular 2-cube $c$ in $\mathbb{R}^7$ for which $\partial c = c_1 - c_2$?

7. Suppose a $k$-dimensional manifold $M$ in $\mathbb{R}^n$ is given near a point $p \in M$ as the zero set of a function $z\colon \mathbb{R}^n \to \mathbb{R}^{n-k}$ whose differential is of maximal rank at $p$, and let $\xi \in T_p\mathbb{R}^n$. Show that $\xi \in T_pM$ if and only if $z_*\xi = 0$. (In doing so, you will have to recall the definition of $T_pM$!)

8. A subset $B$ of $\mathbb{R}^3$ is the union of an infinite line, an infinite ray, and a circle positioned as on the figure below. In addition, oriented loops $R_1, R_2, G_1, G_2$, and $G_3$ are also given as in the same figure. A closed $\omega \in \Omega^1(\mathbb{R}^3 \setminus B)$ is also given, and it is known that $\int_{R_1} \omega = \pi$ and $\int_{R_2} \omega = e$. Compute $\int_{G_i} \omega$ for $i = 1, 2, 3$.




*Hint.* You may want to also think about 2D subsets of $\mathbb{R}^3$ that are shaped like masks, tubes, and/or sacks as in the figure below.



The following questions were a part of a question pool for the 2020-21 MAT257 Final Assessment, but at the end, they were not included.

1. Suppose that the bounded functions $f$ and $g$ are integrable over some rectangle $R \subset \mathbb{R}^n$. Show that $fg$, $f^2$, and $g^2$ are also integrable over $R$ and that $\int_R fg \leq \left(\int_R f^2\right)^{1/2} \left(\int_R g^2\right)^{1/2}$.

2. Prove that the intersection of finitely many open sets in $\mathbb{R}^n$ is open, and give a counterexample to show that this statement may not be true if the intersection is countably infinite.

3. If $A$ and $B$ are disjoint closed sets in $\mathbb{R}^n$, show that there exists disjoint open subsets $C$ and $D$ of $\mathbb{R}^n$ such that $A \subset C$ and $B \subset D$.

4. Recall that the variation $o(f, t)$ of a function $f\colon \mathbb{R} \to \mathbb{R}$ at a point $t \in \mathbb{R}$ is defined to be

$$\lim_{r \to 0} \sup \left\{ |f(x) - f(y)| \colon x, y \in B_r(t) \right\}.$$

Prove that if $f$ is monotone on some interval $[a, b]$ and $P = (a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b)$ is a partition of $[a, b]$, then

$$\sum_{i=1}^{n-1} o(f, t_i) \leq |f(b) - f(a)|.$$

5. Show that the function $f(x, y) = |xy|^{1/2}$ is continuous at $a = (0, 0)$ and has both its partial derivatives exist at $a$, yet it is not differentiable at $a$.

6. Prove that if a differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ has a local max at some point $a \in \mathbb{R}^n$, then $f'(a) = 0$.

7. A continuous function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies $|(x - y) - (f(x) - f(y))| \leq \frac{1}{3}|x - y|$ for every $x, y \in \mathbb{R}^n$. Prove that $f$ is surjective (onto).

8. A function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ has the property that for every $k > 0$ we have $|(x-y)-(f(x)-f(y))| \leq \frac{1}{k}|x-y|$ for every $x, y \in \mathbb{R}^n$ whose norm is less than $e^{-k}$. Prove that $f$ is differentiable at 0 and compute its differential $f'(0)$.

9. A differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}^k$ is said to be "submersive" at 0 if rank $f'(0) = k$. Assume such a function $f$ is submersive at 0 and assume also that $f(0) = 0$, and show that there is a function $g\colon \mathbb{R}^n \to \mathbb{R}^n$ which is defined, differentiable, and invertible near 0, and so that $f(g(x_1, \ldots, x_n)) = (x_1, \ldots, x_k)$. (In other words, every submersive function looks like the standard projection $\mathbb{R}^n \to \mathbb{R}^k$ near 0).

10. (a) A subset $A \subset \mathbb{R}$ is known to have content 0. Is it necessarily true that $\partial A$ also has content 0?
    (b) A subset $B \subset \mathbb{R}$ is known to have measure 0. Is it necessarily true that $\partial B$ also has measure 0?

11. If $f$ is a bounded function defined on a rectangle $R \subset \mathbb{R}^n$ and if supp $f$ (the closure of $\{x \in R\colon f(x) \neq 0\}$) is a set of measure 0, show that $f$ is integrable on $R$ and that $\int_R f = 0$.

12. If $f$ is a bounded non-negative function defined on a rectangle $R \subset \mathbb{R}^n$ and if $\int_R f = 0$, show
    (a) For every $b > 0$, the set $\{x \in R\colon f(x) \geq b\}$ has content 0.
    (b) The set $\{x \in R\colon f(x) > 0\}$ has measure 0.

13. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a smooth function. Prove that there is a smooth function $h\colon \mathbb{R}^2 \to [0, 1]$ such that $h(x, f(x)) = 1$ for every $x \in \mathbb{R}$, yet always, if $x, y \in \mathbb{R}$ and $|y - f(x)| \geq 1$, then $f(x, y) = 0$.

14. Let $m\colon [0, 1] \to M_{n\times n}(\mathbb{R})$ be a path in the space of $n \times n$ matrices, and suppose that for every $t \in [0, 1]$ the columns of $m(t)$ make a basis of $\mathbb{R}^n$. Show that the bases $m(0)$ and $m(1)$ define the same orientation of $\mathbb{R}^n$.

15. Show that if $F = \sum_i f_i(p)(p, e_i)$ and $G = \sum_i G_i(p)(p, e_i)$ are smooth vector fields on $\mathbb{R}^n$, then there is a third smooth vector field $H = \sum_i h_i(p)(p, e_i)$ on $\mathbb{R}^n$ such that

$$D_F \circ D_G - D_G \circ D_F = D_H,$$

where $D_F\colon \Omega^0(\mathbb{R}^n) \to \Omega^0(\mathbb{R}^n)$ is the operation of directional derivative in the direction of $F$, which maps smooth functions on $\mathbb{R}^n$ to smooth functions on $\mathbb{R}^n$ (and likewise for $D_G$ and $D_H$).

16. An exploration problem: a 3-vector on $\mathbb{R}^4_{txyz}$ is a function $F\colon \mathbb{R}^4_{txyx} \to \mathbb{R}^3_{xyz}$. It can be regarded as a time-dependent vector field on $\mathbb{R}^3$, and so it makes sense to write grad, curl, and div in this context, and also $\partial_t = \frac{\partial}{\partial t}$. Of course, you also need to consider "scalar functions" $f\colon \mathbb{R}^4 \to \mathbb{R}$, to talk about grad and div. Can you interpret the sequence

$$\Omega^0(\mathbb{R}^4) \xrightarrow{d} \Omega^1(\mathbb{R}^4) \xrightarrow{d} \Omega^2(\mathbb{R}^4) \xrightarrow{d} \Omega^3(\mathbb{R}^4) \xrightarrow{d} \Omega^4(\mathbb{R}^4)$$

in this language of scalar functions, 3-vectors, grad, curl, div, and $\partial_t$?

17. Prove that the form $xdydz + ydzdx + zdxdy$ is closed but not exact on the 2-dimensional unit sphere $S^2 \subset \mathbb{R}^3_{xyz}$.

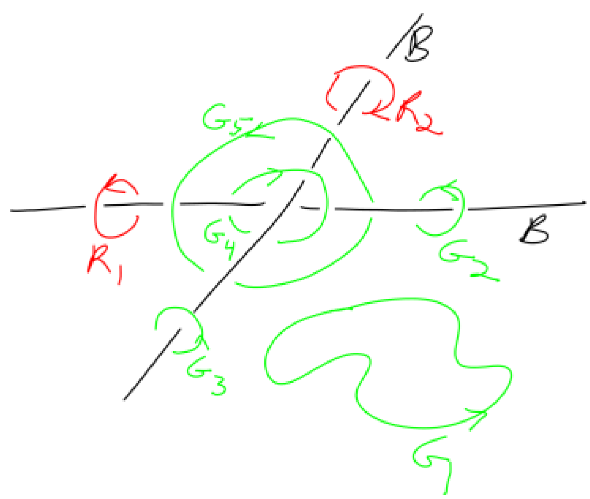
18. $\omega$ is a smooth 3-form on $\mathbb{R}^7$, and we know that the integral of $\omega$ over every 3-cube in $\mathbb{R}^7$ vanishes. Prove that $\omega$ itself vanishes.

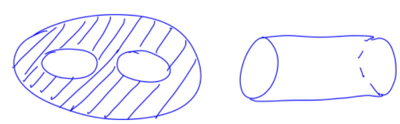
19. We will say that a 1-form $\omega$ on $\mathbb{R}^n$ is "precise" if its integral over any 1-cube depends only on the boundary of that 1-cube (namely, $\partial c_1 = \partial c_2 \Longrightarrow \int_{c_1} \omega = \int_{c_2} \omega$. Show that a 1-form $\omega$ is precise if and only if it is exact.

20. Suppose $M$ is a $k$-dimensional manifold in $\mathbb{R}^n$, and suppose $F$ is a smooth vector field on $M$ (so in particular $F(x) \in T_xM$ for every $x \in M$). Show that there is some vector field $G$ on some open set $A \supset M$ (in particular, $G(x) \in T_x\mathbb{R}^n$ for every $x \in A$) such that $G$ restricted to $M$ is $F$. You may need to use one of the precise definitions of a manifold, and something to make the local go global.

21. A smooth vector field $E$ defined on $\mathbb{R}^3$ is known to satisfy $\operatorname{div} E = 0$ outside of $D^3_{1/2}$, the 3-dimensional closed ball of radius $1/2$ in $\mathbb{R}^3$, and it is also known that $\int_{\partial D^3_1} (E \cdot n)dA = 257$, where everything is taken with "standard conventions": orientations, positive normals, and area forms. Compute $\int_{\partial D^3_2} (E \cdot n)dA$ and $\int_{\partial D^3_1(p)} (E \cdot n)dA$, where $D^3_1(p)$ denotes the closed ball of radius 1 about a point $p$, and $p$ is a point of $\partial D^3_2$.

22. A subset $B$ of $\mathbb{R}^3$ is the union two infinite lines positioned as on the figure on the left below, and in addition, oriented loops $R_1$, $R_2$, and $G_i$ for $i = 1, 2, 3, 4, 5$ are also given as in the same figure. A vector field $F$ is also given, and it is known to be smooth away from $B$ and to satisfy $\operatorname{curl} F = 0$ on $\mathbb{R}^3 \setminus B$. It is known that $\int_{R_1} (F \cdot T)ds = \pi$ and $\int_{R_2} (F \cdot T)ds = e$. Compute $\int_{G_i} (F \cdot T)ds$ for $i = 1, 2, 3, 4, 5$. *Hint.* You may want to also think about 2D subsets of $\mathbb{R}^3$ that are shaped like masks and/or tubes as in the figure below on the right



**Please watch this page for changes — I may add to it later.**

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