Do not open this notebook until instructed.

Math 257 Analysis II

Term Test 3

University of Toronto, March 8, 2022

Solve all 5 problems on this booklet.
The problems are of equal weight.
You have an hour and fifty minutes to write this test.

Notes

- No outside material allowed other than stationery, minimal hydration and snacks, and stuffed animals.
- Write your solution of each problem on the problem page and on the back of the problem page. If you run out of space you may continue into the scratch pages, but you **must** indicate this on the problem page or else the scratch pages will not be read.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and consisting of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.
- **In Red.** Comments added after the test took place.

Good Luck!

Problem 1. In this question, we say that a function $g\colon \mathbb{R}^n \to \mathbb{R}^n$ preserves one coordinate if there is some $k \in \underline{n}$ such that $g_k(x_1, \dots, x_n) = x_k$, where g_k is the kth component function of g. Prove that if $n \geq 2$ and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $f'(0)$ is invertible, then on a neighborhood of 0 we can write $f = g_1 \circ g_2$ where g_1 and g_2 are continuously differentiable functions $\mathbb{R}^n \to \mathbb{R}^n$ and each preserves one coordinate.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Tip. You may want to start by writing "draft solutions" on the last pages of this notebook and only then write the perfected versions in the space left here for solutions.

Ouch. This problem is buggy. During the test I added the necessary assumption $\exists i\ \frac{\partial f_i(0)}{\partial x_i} \neq 0$. Some further comments:

- The functions g_1 and g_2 need only be defined on a neighborhood of 0, and not on all of $\mathbb{R}^n$ as was written. I did not make this comment during the test because it seemed minor and clear enough, and no points will be deducted if you completely ignored this issue.
- The naming g_1 and g_2 conflicts with the notation for the components of a function, g_k. Strictly sppeaking, this is not a bug. It's just a point of inelegance. Sorry.
- With the new assumption the condition that $f'(0)$ is invertible is no longer needed.

Problem 2. Let $\phi\colon \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{u,v}$ be given by $\phi(x,y) = (e^x \cos y, e^x \sin y)$. Compute $\phi^*(du \wedge dv)$ and $\phi_*\xi$, where ξ is the tangent vector to $\mathbb{R}^2_{x,y}$ given by $\xi = \left(\begin{pmatrix} 0 \\ \pi/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$.

Problem 3. Let V be a vector space, let $\phi\colon V \to V \times V$ be given by $\phi(v) = (v,v)$ and let $\psi\colon V \times V \to V \times V$ be given by $\psi(v,w) = (w,v)$. Let $B\colon V \times V \to \mathbb{R}$ be a bilinear function. Prove that $\phi^*B = 0$ iff $B + \psi^*B = 0$.

Problem 4. Explain in detail how the vector field operator curl arises as an instance of the exterior derivative operator $d\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$, for some k and n.

Reminder. $\operatorname{curl}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}$.

Problem 5. If $L\colon V \to W$ is an invertible linear transformation between oriented vector spaces (vector spaces equipped with an orientation), we say that L is *orientation preserving* if it pushes the orientation of V forward to the orientation of W (or equivalently, if it pulls the orientation of W back to the orientation of V). Otherwise, L is called *orientation reversing*. Decide for each of the cases below, if L_i is orientation preserving or reversing. In this question $\mathbb{R}^n$ always comes equipped with its standard orientation $(e_1, e_2, \dots, e_n)$.

1. $L_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ via $(x,y) \mapsto (-x,y)$.
2. $L_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ via $(x,y) \mapsto (y,x)$.
3. $L_3\colon \mathbb{R}^2 \to \mathbb{R}^2$, the counterclockwise rotation by $2\pi/7$.
4. $L_4\colon \mathbb{R}^2 \to \mathbb{R}^2$, the clockwise rotation by $2\pi/7$.
5. $L_5\colon \mathbb{R}^2 \to \mathbb{R}^2$, the complex conjugation map $z \mapsto \bar{z}$, where $\mathbb{R}^2$ is identified with $\mathbb{C}$ via $(x,y) \leftrightarrow x + iy$.
6. $L_6\colon \mathbb{R}^3 \to \mathbb{R}^3$ via $(x,y,z) \mapsto (y,z,x)$.
7. $L_7\colon \mathbb{R}^n \to \mathbb{R}^n$ via $v \mapsto -v$.

8. $L_8 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ via $(u, v) \mapsto (v, u)$, where $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$.

Tip. The answers for L_7 and for L_8 may depend on n and m.