Pensieve header: The category GDO and the Heisenberg algebra. Based on AlexanderFromHeisenberg.nb at pensieve://Talks/LearningSeminarOnCategorification-2006/.

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Classes\\21-1350-KnotTheory"];
<< Common.m
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

A hack to fix how Expand acts on Series:

In[∘]:=
```
Unprotect[SeriesData];
SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
Protect[SeriesData];
```

In[∘]:=
```
{p*, x*, π*, ξ*} = {π, ξ, p, x}; (u_i_)* := (u*)_i; l_List* := #* & /@ l;
```

In[∘]:=
```
(𝔾_{A1_→B1_}[L1_] // 𝔾_{A2_→B2_}[L2_]) /; (B1* === A2) := Module[{b, r, x, ξ},
  r = Expand[(L1 /. Table[x → b[x], {x, B1}]) (L2 /. Table[ξ → b[ξ], {ξ, A2}])];
  Do[r = r /. b[x]^p_. b[x*]^p_. → p! /. b[x | x*] → 0, {x, B1}];
  𝔾_{A1→B2}[r]
 ]
```

In[∘]:= $\mathbb{G}_{\{\xi_1\}\to\{x_1\}}\left[1+\xi_1 x_1+\xi_1^2 x_1^2\right]$ // $\mathbb{G}_{\{\xi_1\}\to\{x_2\}}\left[1+\xi_1 x_2+\xi_1^2 x_2^2\right]$

Out[∘]= $\mathbb{G}_{\{\xi_1\}\to\{x_2\}}\left[1+x_2\,\xi_1+2\,x_2^2\,\xi_1^2\right]$

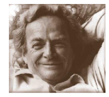
GDO := The category with objects finite sets and

$$\mathrm{mor}(A\to B)=\left\{\mathcal{L}=\omega\mathbb{e}^{Q}\right\}\subset\mathbb{Q}[\![\zeta_A,z_B]\!],$$

where: • ω is a scalar. • Q is a “small” quadratic in $\zeta_A\cup z_B$.
• Compositions: $\mathcal{L}/\!\!/\mathcal{M}:=\left(\mathcal{L}|_{z_i\to\partial_{\zeta_i}}\mathcal{M}\right)_{\zeta_i=0}$.

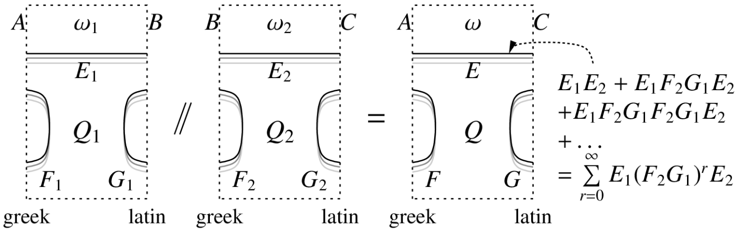
Compositions. In mor$(A \to B)$,

$$Q = \sum_{i\in A, j\in B} E_{ij}\zeta_i z_j + \frac{1}{2}\sum_{i,j\in A} F_{ij}\zeta_i\zeta_j + \frac{1}{2}\sum_{i,j\in B} G_{ij}z_i z_j,$$



R. Feynman

and so (remember, $e^x = 1 + x + xx/2 + xxx/6 + \ldots$)



where • $E = E_1(I-F_2G_1)^{-1}E_2$ • $F = F_1 + E_1F_2(I-G_1F_2)^{-1}E_1^T$
• $G = G_2 + E_2^T G_1(I-F_2G_1)^{-1}E_2$ • $\omega = \omega_1\omega_2 \det(I-F_2G_1)^{-1/2}$

In[∘]:=
```
𝔼s1_[ωQ1__] ≡ 𝔼s2_[ωQ2__] := s1 === s2 ∧ Simplify[{ωQ1} == {ωQ2}]
```

In[∘]:=
```
𝔼A1_→B1_[ω1_, Q1_] 𝔼A2_→B2_[ω2_, Q2_] ^:= 𝔼A1⋃A2→B1⋃B2[ω1 ω2, Q1 + Q2]
```

In[∘]:=
```
CF = ExpandNumerator@*ExpandDenominator@*PowerExpand@*Factor;
```

In[∘]:=
```
(𝔼A1_→B1_[ω1_, Q1_] // 𝔼A2_→B2_[ω2_, Q2_]) /; (B1* === A2) :=
 Module[{i, j, E1, F1, G1, E2, F2, G2, I, M = Table},
  I = IdentityMatrix@Length@B1;
  E1 = M[∂i,j Q1, {i, A1}, {j, B1}]; E2 = M[∂i,j Q2, {i, A2}, {j, B2}];
  F1 = M[∂i,j Q1, {i, A1}, {j, A1}]; F2 = M[∂i,j Q2, {i, A2}, {j, A2}];
  G1 = M[∂i,j Q1, {i, B1}, {j, B1}]; G2 = M[∂i,j Q2, {i, B2}, {j, B2}];
  𝔼A1→B2[CF[ω1 ω2 Det[I - F2.G1]^(-1/2)], CF@Plus[
     If[A1 === {} ∨ B2 === {}, 0, A1.E1.Inverse[I - F2.G1].E2.B2],
     If[A1 === {}, 0, 1/2 A1.(F1 + E1.F2.Inverse[I - G1.F2].E1ᵀ).A1],
     If[B2 === {}, 0, 1/2 B2.(G2 + E2ᵀ.G1.Inverse[I - F2.G1].E2).B2]]]]
```

In[∘]:=
```
Random𝔼A_→B_[r_] := Module[{ri},
  ri := RandomInteger[{-r, r}];
  𝔼Table[ξi,{i,A}]→Table[xj,{j,B}][
   ri,
   Sum[ri ℏ ξi ξj, {i, A}, {j, A}] +
    Sum[ri ℏ ξi xj, {i, A}, {j, B}] + Sum[ri ℏ xi xj, {i, B}, {j, B}]
  ]]
```

In[∘]:= **E1 = Random𝔼$_{\{1,2\}\to\{1,2,3,4\}}$[5]**

Out[∘]= $$\mathbb{E}_{\{\xi_1,\xi_2\}\to\{x_1,x_2,x_3,x_4\}}\left[0,\ -5\hbar x_1^2+6\hbar x_1 x_2-\hbar x_1 x_3-\hbar x_2 x_3+3\hbar x_3^2-6\hbar x_1 x_4+7\hbar x_2 x_4-2\hbar x_3 x_4-2\hbar x_4^2-\hbar x_1\xi_1+4\hbar x_2\xi_1+5\hbar x_3\xi_1+5\hbar x_4\xi_1+5\hbar\xi_1^2-2\hbar x_1\xi_2+2\hbar x_3\xi_2+\hbar x_4\xi_2-3\hbar\xi_2^2\right]$$

In[∘]:= **E2 = Random𝔼$_{\{1,2,3,4\}\to\{1,2,3\}}$[5]**

Out[∘]= $$\mathbb{E}_{\{\xi_1,\xi_2,\xi_3,\xi_4\}\to\{x_1,x_2,x_3\}}\left[5,\ 5\hbar x_1^2-4\hbar x_1 x_2+2\hbar x_2^2-6\hbar x_1 x_3-5\hbar x_2 x_3+2\hbar x_3^2-4\hbar x_1\xi_1-5\hbar x_2\xi_1+\hbar x_3\xi_1+\hbar\xi_1^2+5\hbar x_1\xi_2-\hbar x_2\xi_2+2\hbar x_3\xi_2+4\hbar\xi_1\xi_2-\hbar\xi_2^2+4\hbar x_1\xi_3-5\hbar x_2\xi_3-2\hbar x_3\xi_3-\hbar\xi_1\xi_3-8\hbar\xi_2\xi_3-5\hbar\xi_3^2+2\hbar x_1\xi_4+3\hbar x_2\xi_4-\hbar x_3\xi_4+\hbar\xi_1\xi_4-5\hbar\xi_2\xi_4-3\hbar\xi_3\xi_4-5\hbar\xi_4^2\right]$$

In[∘]:= **E1 // E2**

Out[∘]= $$\mathbb{E}_{\{\xi_1,\xi_2\}\to\{x_1,x_2,x_3\}}\Big[0,\ \big(-5\hbar x_1^2-158\hbar^3 x_1^2+14789\hbar^5 x_1^2+54793\hbar^7 x_1^2+8020141\hbar^9 x_1^2+4\hbar x_1 x_2+398\hbar^3 x_1 x_2+2961\hbar^5 x_1 x_2+350502\hbar^7 x_1 x_2+6794793\hbar^9 x_1 x_2-2\hbar x_2^2-119\hbar^3 x_2^2+3837\hbar^5 x_2^2+17530\hbar^7 x_2^2+6709271\hbar^9 x_2^2+6\hbar x_1 x_3+319\hbar^3 x_1 x_3-1430\hbar^5 x_1 x_3+311861\hbar^7 x_1 x_3-192074\hbar^9 x_1 x_3+5\hbar x_2 x_3+158\hbar^3 x_2 x_3-4457\hbar^5 x_2 x_3+282598\hbar^7 x_2 x_3+872044\hbar^9 x_2 x_3-2\hbar x_3^2-99\hbar^3 x_3^2+2440\hbar^5 x_3^2+39671\hbar^7 x_3^2+1127909\hbar^9 x_3^2-54\hbar^2 x_1\xi_1+1312\hbar^4 x_1\xi_1+39676\hbar^6 x_1\xi_1-281514\hbar^8 x_1\xi_1+9\hbar^2 x_2\xi_1-1420\hbar^4 x_2\xi_1+98513\hbar^6 x_2\xi_1-1047796\hbar^8 x_2\xi_1+8\hbar^2 x_3\xi_1+417\hbar^4 x_3\xi_1+32722\hbar^6 x_3\xi_1-701097\hbar^8 x_3\xi_1-5\hbar\xi_1^2+396\hbar^3\xi_1^2-4632\hbar^5\xi_1^2-24391\hbar^7\xi_1^2+45760\hbar^9\xi_1^2-18\hbar^2 x_1\xi_2+245\hbar^4 x_1\xi_2+9494\hbar^6 x_1\xi_2-712723\hbar^8 x_1\xi_2-3\hbar^2 x_2\xi_2-345\hbar^4 x_2\xi_2+38259\hbar^6 x_2\xi_2-1042657\hbar^8 x_2\xi_2+7\hbar^2 x_3\xi_2+402\hbar^4 x_3\xi_2+2415\hbar^6 x_3\xi_2-492844\hbar^8 x_3\xi_2+306\hbar^3\xi_1\xi_2-7451\hbar^5\xi_1\xi_2-23034\hbar^7\xi_1\xi_2+674245\hbar^9\xi_1\xi_2+3\hbar\xi_2^2+157\hbar^3\xi_2^2-4778\hbar^5\xi_2^2+53471\hbar^7\xi_2^2-878020\hbar^9\xi_2^2\big)\Big/\big(-1-44\hbar^2+938\hbar^4-15840\hbar^6+501215\hbar^8\big)\Big]$$

In[∘]:=
```
𝔼Series[𝔼s_[ω_, Q_], d_] := 𝔾s[Expand@Series[ω Exp[Q], {ℏ, 0, d}]]
```

In[∘]:= **𝔼Series[E1, 1]**

Out[∘]= $\mathbb{G}_{\{\xi_1,\xi_2\}\to\{x_1,x_2,x_3,x_4\}}[0]$

In[∘]:= **𝔼Series[E2, 1]**

Out[∘]= $$\mathbb{G}_{\{\xi_1,\xi_2,\xi_3,\xi_4\}\to\{x_1,x_2,x_3\}}\Big[5+\big(25x_1^2-20x_1x_2+10x_2^2-30x_1x_3-25x_2x_3+10x_3^2-20x_1\xi_1-25x_2\xi_1+5x_3\xi_1+5\xi_1^2+25x_1\xi_2-5x_2\xi_2+10x_3\xi_2+20\xi_1\xi_2-5\xi_2^2+20x_1\xi_3-25x_2\xi_3-10x_3\xi_3-5\xi_1\xi_3-40\xi_2\xi_3-25\xi_3^2+10x_1\xi_4+15x_2\xi_4-5x_3\xi_4+5\xi_1\xi_4-25\xi_2\xi_4-15\xi_3\xi_4-25\xi_4^2\big)\hbar+O[\hbar]^2\Big]$$

In[∘]:= **𝔼Series[E1, 1] // 𝔼Series[E2, 1]**

Out[∘]= $\mathbb{G}_{\{\xi_1,\xi_2\}\to\{x_1,x_2,x_3\}}[0]$

In[∘]:= **𝔼Series[E1 // E2, 1]**

Out[∘]= $\mathbb{G}_{\{\xi_1,\xi_2\}\to\{x_1,x_2,x_3\}}[0]$

In[∘]:= **𝔼Series[E1 // E2, 5] == (𝔼Series[E1, 5] // 𝔼Series[E2, 5])**

Out[∘]= True

In[◦]:=
$$A\_ \setminus B\_ := \text{Complement}[A, B];$$
$$\left(\mathbb{E}_{A1\_\to B1\_}[\omega 1\_, Q1\_] \mathbin{//} \mathbb{E}_{A2\_\to B2\_}[\omega 2\_, Q2\_]\right) \text{ /; } (B1^* \mathrel{=!=} A2) :=$$
$$\mathbb{E}_{A1\cup(A2\setminus B1^*)\to B1\cup A2^*}[\omega 1, Q1 + \text{Sum}[\zeta^*\,\zeta, \{\zeta, A2\setminus B1^*\}]] \mathbin{//}$$
$$\mathbb{E}_{B1^*\cup A2\to B2\cup(B1\setminus A2^*)}[\omega 2, Q2 + \text{Sum}[z^*\,z, \{z, B1\setminus A2^*\}]]$$

A proof of the formula for *R* is at http://drorbn.net/cat20.

In[◦]:=
$$R_{i\_,j\_} := \mathbb{E}_{\{\}\to\{p_i,x_i,p_j,x_j\}}\left[T^{1/2}, (1-T)\,p_j\,x_j + (T-1)\,p_i\,x_i\right];$$
$$\overline{R}_{i\_,j\_} := \mathbb{E}_{\{\}\to\{p_i,x_i,p_j,x_j\}}\left[T^{-1/2}, \left(1-T^{-1}\right)p_j\,x_j + \left(T^{-1}-1\right)p_i\,x_i\right];$$
$$C_{i\_} := \mathbb{E}_{\{\}\to\{p_i,x_i\}}\left[T^{1/2}, 0\right];\quad \overline{C}_{i\_} := \mathbb{E}_{\{\}\to\{p_i,x_i\}}\left[T^{-1/2}, 0\right];$$

In[◦]:=
$$\eta_{i\_} := \mathbb{E}_{\{\}\to\{p_i,x_i\}}[1, 0]$$

In[◦]:=
$$m_{i\_,j\_\to k\_} := \mathbb{E}_{\{\pi_i,\xi_i,\pi_j,\xi_j\}\to\{p_k,x_k\}}\left[1, -\xi_i\,\pi_j + \left(\pi_i+\pi_j\right)p_k + \left(\xi_i+\xi_j\right)x_k\right]$$

Reidemeister 3

In[◦]:=
$$(R_{1,2}\,R_{4,3}\,R_{5,6} \mathbin{//} m_{1,4\to1}\,m_{2,5\to2}\,m_{3,6\to3}) \equiv (R_{2,3}\,R_{1,6}\,R_{4,5} \mathbin{//} m_{1,4\to1}\,m_{2,5\to2}\,m_{3,6\to3})$$

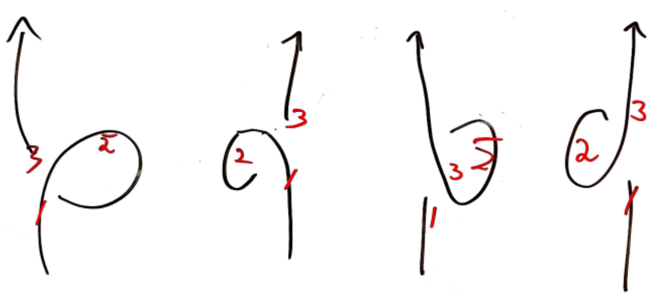
Out[◦]= True

Reidemeister 2

In[◦]:=
$$\left\{\left(\overline{R}_{1,2}\,R_{3,4} \mathbin{//} m_{1,3\to1}\,m_{2,4\to2}\right) \equiv \eta_1\,\eta_2,\right.$$
$$\left.\left(R_{1,4}\,\overline{R}_{5,2}\,\overline{C}_3 \mathbin{//} m_{2,4\to2} \mathbin{//} m_{1,3\to1} \mathbin{//} m_{1,5\to1}\right) \equiv \overline{C}_1\,\eta_2\right\}$$

Out[◦]= {True, True}

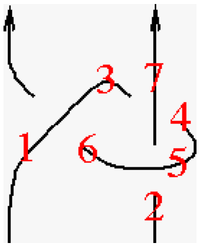
Reidemeister 1’s



In[◦]:=
$$\left\{\left(\overline{C}_2\,R_{1,3} \mathbin{//} m_{1,2\to1} \mathbin{//} m_{1,3\to1}\right) \equiv \eta_1,\ \left(\overline{C}_2\,\overline{R}_{3,1} \mathbin{//} m_{1,2\to1} \mathbin{//} m_{1,3\to1}\right) \equiv \eta_1,\right.$$
$$\left.\left(C_2\,\overline{R}_{1,3} \mathbin{//} m_{1,2\to1} \mathbin{//} m_{1,3\to1}\right) \equiv \eta_1,\ (C_2\,R_{3,1} \mathbin{//} m_{1,2\to1} \mathbin{//} m_{1,3\to1}) \equiv \eta_1\right\}$$

Out[◦]= {True, True, True, True}

The “First Tangle”



In[∘]:= **Factor /@ (z = $R_{1,6}\,\overline{C}_3\,\overline{R}_{7,4}\,\overline{R}_{5,2}$ // $m_{1,3\to 1}$ // $m_{1,4\to 1}$ // $m_{1,5\to 1}$ // $m_{1,6\to 1}$ // $m_{2,7\to 2}$)**

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}\left[\frac{T}{-1+2T}, \frac{(-1+T)\,(p_1-p_2)\,(T x_1-x_2)}{-1+2T}\right]$

In[∘]:= **ZF[Knot[4, 1]]**

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{T}{1-3T+T^2}, 0\right]$

In[∘]:= **Alexander[Knot[4, 1]][T]**

Out[∘]= $3-\frac{1}{T}-T$

In[∘]:= **Timing@Table[Simplify[Alexander[K][T] × ZF[K]⟦1⟧], {K, AllKnots[{3, 9}]}]**

Out[∘]= {28.1719, {1, -1, 1, 1, -1, -1, 1, 1, 1, 1, 1, 1, -1, 1, -1, -1, -1, -1, -1, -1, 1, 1, -1, 1, -1, 1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, 1, 1, 1, 1, 1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, 1, 1, 1, -1, -1, 1, 1, -1, -1, 1, -1, 1, 1, -1, 1, 1, -1, -1, 1, -1, 1, 1, -1, 1, 1, -1, -1, 1, -1, -1, 1, -1, 1}}

In[∘]:= **Timing@ZF[GST48]**

Out[∘]= $\left\{49.3125, \mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{T^8}{1-2T+T^2+2T^4-5T^5+2T^6+7T^7-13T^8+7T^9+2T^{10}-5T^{11}+2T^{12}+T^{14}-2T^{15}+T^{16}}, 0\right]\right\}$