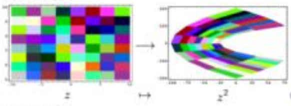
240AlgebraI-141029 Hours 22-23: isomorphisms, rank-nullity, matrices

October-21-14 1:36 PM

14-240/Linear Algebra - Why We Care

Linear Algebra is the Small Scale Theory of *Everything* [edit]

- **To study the large, start with the small.**
- **In small scales, every space is a vector space.**
  - Indeed if you walk a mile east, a mile north, a mile west and a mile south, you're back were you started, but if you fly a 1,000 miles east, a 1,000 miles north, a 1,000 miles west and a 1,000 miles south, you're not back were you started (where will you be?).
- **In small scales, every function is a linear function.**



- **The world doesn't come with coordinates.**
  - Hence whenever we can we work without a basis, and when we do study bases, we study all of them.

See also The Mathematica notebook QuiltPlot.

Read Along 2.1–2.3

Today: isomorphisms, rank nullity, matrices.

Reminder. A l.t. is determined by its values on a basis, and these values are arbitrary.

Meaning: Given V, basis $\beta=(u_1 \dots u_n)$, W, sequence $(w_1 \dots w_n)$, $\exists!\, L: V \to W$ s.t. $L(u_i)=w_i$

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start class w/ "Why we care"!

Two mathematical structures are "isomorphic" if there's a bijection (=: 1–1 & onto map / invertible map) between their elements which preserves all relevant relations.

Example plastic chess is iso. to ivory chess, but not to checkers.

Example The game of 15.

Def V & W are isomorphic if $\exists$ l.t. $R: V \to W$ and $L: W \to V$ s.t. $L \circ R = I_V$ & $R \circ L = I_W$

Thm If V, W are f.d. over F, then $\dim V = \dim W$ iff V is isomorphic to W.

Proof.....

Corollary If $\dim V = n$ over F, V is isomorphic to F^n.

Given $(V, \underset{(u_1 \dots u_n)}{\beta})$, get iso $V \to F^n$: $V \ni x \mapsto [x]_\beta \in F^n$

$$[x_\beta] = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \iff x = \sum \alpha_i u_i$$

Example $[(x-1)^3]_{1, x, x^2, x^3} = \begin{pmatrix} -1 \\ 3 \\ -3 \\ 1 \end{pmatrix}$

Fix a l.t. $T: V \to W$

Def $N(T) = \ker T = \{v : Tv = 0\}$ "null space", "kernel".

$R(T) = \operatorname{im} T = \{Tv : v \in V\}$ "range", "image"

Prop/Def $N(T) \subset V$ is a subspace; nullity$(T) := \dim N(T)$

placement is awkward — it's a coord.-free statement

$R(T) = \operatorname{im} T = \{Tv : v \in V\}$ "range", "image"

— it's a coord.-free statement placed after the discussion of coordinates had already begun.

<u>Prop/Def</u> $N(T) \subset V$ is a subspace; $\text{nullity}(T) := \dim N(T)$

$R(T) \subset W$ is a subspace; $\text{rank}(T) := \dim R(T)$

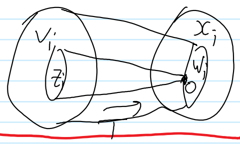
<u>Examples</u> 0, I_V, $D : P_n(\mathbb{R}) \to P_n(\mathbb{R})$

<u>Thm 1</u> "the dimension theorem", "the rank-nullity Thm"

Given $T : V \to W$, $\underset{m}{\dim V} = \underset{r}{\text{rank}(T)} + \underset{n}{\text{nullity}(T)}$

<u>PF</u> $(z_i)_1^n$ basis of $N(T)$, extend to $(z_i) \cup (v_i)$ a basis of V,

done line.



<u>claim</u> $w_i := T(v_i)$ are lin indep. in W pf....

<u>claim</u> w_i span $R(T)$ pf...

<u>Corollary of thm 1.</u> If $\dim V = \dim W$, TFAE

1. T is 1-1   2. T is onto
3. $\text{rank}\, T = \dim V$ 4. T is invertible.

<u>Thm 2</u> $T : V \to W$ & $T' : V' \to W'$ are "isomorphic" iff $(\dim V, \dim W, \text{rank}\, T) = (\dim V', \dim W', \text{rank}\, T')$

i.e., $\exists$ a "commutative square of isomorphisms":

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \phi\downarrow & & \downarrow\psi \\ V' & \xrightarrow{T'} & W' \end{array}$$

skipable.

<u>Reminder</u>: Choosing a basis, V is isomorphic to F^n.

<u>Goal</u>: Choosing bases, $\mathcal{L}(V, W)$ is isomorphic to $M_{m\times n}$ ($m = \dim W$, $n = \dim V$)

<u>Thm.</u> Given V w/ basis $\beta = (v_1 \ldots v_n)$ and W w/ basis $\gamma = (w_1 \ldots w_m)$

We have an isomorphism

Abstract, general, coord-free   merely numbers, choice-dependent, easy to work with

$$\mathcal{L}(V, W) \longrightarrow M_{m\times n}(F)$$

$$T \longrightarrow [T]_\beta^\gamma = A$$

$$A = \begin{pmatrix} a_{11} & & & a_{1n} \\ \vdots & & & \vdots \\ [Tv_1]_\gamma & [Tv_2]_\gamma & \cdots & [Tv_n]_\gamma \\ \vdots & & & \vdots \\ a_{m1} & & & a_{mn} \end{pmatrix} \iff T v_j = \sum_{i=1}^{m} a_{k,j} w_k$$

Examples 0. 0 1. 1

2. $D : P_3(\mathbb{R}) \longrightarrow P_2(\mathbb{R})$ differentiation

3. $T_\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

4. $A : F^n \longrightarrow F^m$

Added Nov 10, 2014: I should have added:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[\ ]_\beta}"'] & W \arrow[d, "{[\ ]_\gamma}"] \\ F^n \arrow[r, "A", "T_A"'] & F^m \end{tikzcd}$$