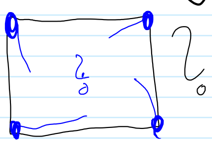
240AlgebraI-141008 Hours 14-15: linear independence, bases

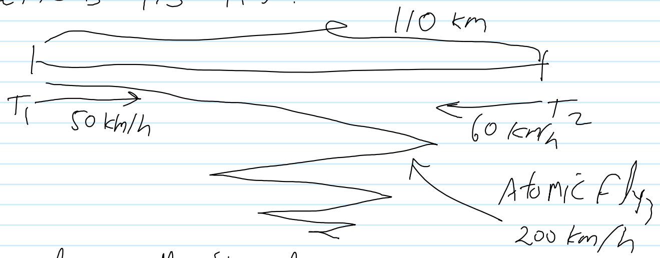
September-18-14 6:29 PM

Read Along sections 1.5–1.7.

post mortem: I failed to organize this class as "telling a story".

Riddle Along





How long will it fly before crashing?

Today: lin. independence, bases. (our first *real* thm)

Reminder. S "spans"/"generates"

$\iff \forall u \in V\ \exists \alpha_i \in F,\ u_i \in S$ s.t. $u = \sum \alpha_i u_i$

S "linearly independent" $\iff$

$\left(\sum \alpha_i u_i = 0,\ u_i \in S \text{ distinct} \Rightarrow \forall i\ \alpha_i = 0\right)$

Comments 1. $\emptyset$ is lin. indep.

2. $\{u\}$ is lin indep iff $u \neq 0$.

3. Suppose $S_1 \subset S_2 \subset V$. Then

a. If S_1 is dep, so is S_2

b. If S_2 is indep, so is S_1

4. If S is lin indep in V and $v \in V$, then $S \cup \{v\}$ is lin. dep. iff $v \in \operatorname{span}(S)$.

Def Basis $\beta \subset V$

Thm A subset $\beta \subset V$ is a basis iff every $v \in V$ can be expressed in a unique way as a l.c. of elements of β

I should have stated an proven this merely for ordered bases.

Thm If a finite set S generates a V.S. V, then there is a subset $\beta \subseteq S$ which is a basis of V

Pf Let β be a lin indep subset of S which is of maximal size. Then every $v \in S \setminus \beta$ satisfies $v \in \operatorname{span} \beta$, so $S \subset \operatorname{span} \beta$, so $\operatorname{span} S \subset \operatorname{span} \beta$.

Our first non-language Theorem:

Thm If a v.s. V has a finite basis, then every other basis of V has the same number of elements in it.

Def If V has a finite basis, we say that it is "finite-dimensional" and let

$$\dim V := \begin{pmatrix} \text{The number of elements} \\ \text{in (any) basis of } V \end{pmatrix}$$

Examples as before.

$\{0\}$, F^n, $M_{m\times n}$, $P_n(F)$, $P(F)$

done

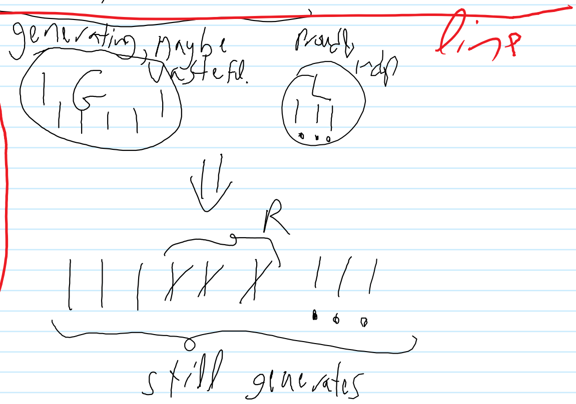
Lemma (the replacement lemma)

$\operatorname{span} G = V$ $\quad$ L lin indep w/ $|L| = n$

$\Longrightarrow \exists R \subset G$ with

$|R| = n$ and $\operatorname{span}((G \setminus R) \cup L) = V$

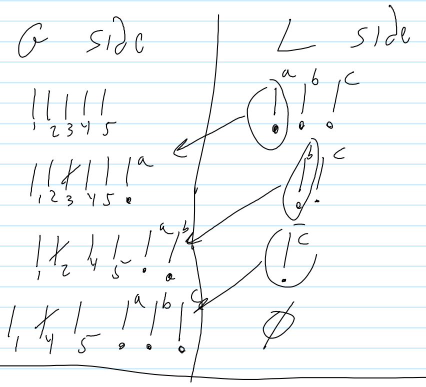
in particular, $|L| \le |G|$



Pf of theorem from lemma.

Informal proof of Lemma First of all, if $\sum a_i u_i = 0$, the any vector that appears in this dependency with non-zero coeff is a l.c. of the others.

Now:



Formal proof: Induction on $|L|$. $|L| = 0$: trivial.

Now $|L| = n+1$; $L = \{v_1 \ldots v_{n+1}\}$. Use $L' = \{v_1 \ldots v_n\}$,

Find $R' = \{r_1 \ldots r_n\} \subset G$ s.t. $(G \setminus \{r_1 \ldots r_n\}) \cup \{v_1 \ldots v_n\}$ spans. write

$$v_{n+1} = \sum^{m} a_i u_i + b_1 v_1 + \ldots + b_n v_n \qquad u_i \in G \setminus \{r_1 \ldots r_n\}$$

$\therefore$ Not all $a_i = 0$; let j be such that $a_j \neq 0$

$\therefore$ so $u_j \in \operatorname{span}(u_1 \cancel{u_j}\, u_m\ v_1 \dots v_{m+1})$,

so take $r_{n+1} = u_j$ & $R = R' \cup \{r_{n+1}\}$.

Corollaries: 1. If V has a finite basis β_1 then every other basis β_2 of V is also finite & $|\beta_1| = |\beta_2|$.

2. "$\dim V$" makes sense.

3. Assume $\dim V = n$. Then

a. If G generates V, $|G| \geq n$ & if also $|G| = n$, then G is a basis.

b. If L is linearly indep in V, the $|L| \leq n$;
if also $|L| = n$, L is a basis.
if also $|L| < n$, L can be extended to a basis.

4. If V is finite-dimensional and $W \subset V$ is a subspace, then W is f.d. and $\dim W \leq \dim V$.
If also $\dim W = \dim V$, then $W = V$.
If also $\dim W < \dim V$, then any basis of W can be extended to a basis of V.