Do not turn this page until instructed.

Math 240 Algebra I

Term Test

University of Toronto, October 21, 2014

Solve 4 of the 5 problems on the other side of this page.
Each problem is worth 25 points.
You have an hour and fifty minutes to write this test.

Notes

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.

A *Field* is a set $\mathbb{F}$ along with two binary operations $+, \times\colon \mathbb{F}\times\mathbb{F} \to \mathbb{F}$ and two distinguished elements $0 \neq 1 \in \mathbb{F}$ such that: **F1.** $\forall a, b \in \mathbb{F}\colon\ [a + b = b + a] \wedge [ab = ba]$. **F2.** $\forall a, b, c \in \mathbb{F}\colon\ [(a + b) + c = a + (b + c)] \wedge [(ab)c = a(bc)]$. **F3.** $\forall a \in \mathbb{F}\colon\ [a + 0 = a] \wedge [a \cdot 1 = a]$. **F4.** $\forall a \in \mathbb{F}\, \exists b \in \mathbb{F}\colon\ a + b = 0$ and $\forall a \in \mathbb{F}\colon\ (a \neq 0) \Rightarrow [\exists b \in \mathbb{F}\colon\ ab = 1]$. **F5.** $\forall a, b, c \in \mathbb{F}\colon\ (a + b)c = ac + bc$.

A *Vector Space* over a field $\mathbb{F}$ is a set $\mathbb{V}$ along with two binary operations $+\colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ and $\times\colon \mathbb{F} \times \mathbb{V} \to \mathbb{V}$ and a distinguished element $0 \in \mathbb{V}$ such that: **VS1.** $\forall x, y \in \mathbb{V}\colon\ x + y = y + x$. **VS2.** $\forall x, y, z \in \mathbb{V}\colon\ (x + y) + z = x + (y + z)$. **VS3.** $\forall x \in \mathbb{V}\colon\ x + 0 = x$. **VS4.** $\forall x \in \mathbb{V}\, \exists y \in \mathbb{V}\colon\ x + y = 0$. **VS5.** $\forall x \in \mathbb{V}\colon\ 1x = x$. **VS6.** $\forall a, b \in \mathbb{F}\, \forall x \in \mathbb{V}\colon\ a(bx) = (ab)x$. **VS7.** $\forall a \in \mathbb{F}\, \forall x, y \in \mathbb{V}\colon\ a(x + y) = ax + ay$. **VS8.** $\forall a, b \in \mathbb{F}\, \forall x \in \mathbb{V}\colon\ (a + b)x = ax + bx$.

From Project Gutenberg's Cautionary Tales for Children, by Hilaire Belloc

http://www.gutenberg.org/files/27424/27424-h/27424-h.htm

Good Luck!

Solve 4 of the following 5 problems. Each problem is worth 25 points. You have an hour and fifty minutes. **Neatness counts! Language counts!**

Problem 1. Let V be a vector space over a field F, let $c \in F$ and let $v \in V$.

1. Prove that if $v = 0$ then $cv = 0$.
2. Prove that if $cv = 0$, then either $c = 0$ or $v = 0$.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Problem 2.

1. In the field $\mathbb{C}$ of complex numbers, compute

$$4(2i - 1)(1 + i) \qquad \text{and} \qquad \frac{4i}{1+i}.$$

   (To be precise, "compute" means "write in the form $a + bi$, where $a, b \in \mathbb{R}$").

2. Working in the 7-element field F_7 of remainders modulo 7, make a table showing the values of a^{-1} for every $a \neq 0$.

Problem 3. Let V be a vector space and let W_1 and W_2 be subspaces of V. Prove that $W_1 \cup W_2$ is a subspace of V iff $W_1 \subset W_2$ or $W_2 \subset W_1$.
Tip. "If and only if" always means that there are two things to prove.

Problem 4. In the vector space $P_3(\mathbb{Q})$ of polynomials of degree at most 3 with rational coefficients, decide whether the polynomial $p = x^3 + 2x^2 - 3x + 4$ is a linear combination of the polynomials $u_1 = x^3 - x$, $u_2 = x^2 + 1$, and $u_3 = x^3 + x^2$.
Tip. It is always an excellent idea to substitute your solutions back into the equations and see if they really work.

Problem 5. State and prove the "replacement lemma".
Tip. In math-talk, "state" means "write the statement of, in full". Yet then, don't forget to also prove!

Good Luck!