# Wednesday-3 AKT on 140122: More on swaddling maps and on framings

January-16-14 4:38 AM

Last Wednesday: In the discussion of degrees the two manifolds involved must be "closed", meaning compact and having no boundary.

Class photo today!

## Invariants in $\mathbb{Z}/\mathbb{Z}$!

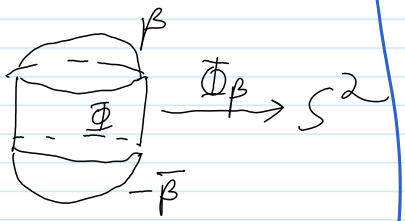
$$\eta(\gamma) := \frac{1}{4\pi}\int_{C_2(S^1)} \Phi^*\omega = \frac{1}{4\pi}\int_{\tilde{C}_2(S_1)} \Phi^*\omega$$

$$C_2(S^1) = \{(x,y) : x,y \in S^1,\ x \neq y\} = S^1_x \times (0,1)_z \quad y = x+z$$

$$\longrightarrow \tilde{C}_2(S^1) = S^1 \times [0,1]$$

choose a "swaddling map" $\beta : D^2 \to S^2$

s.t. $\beta|_{S^1} = \gamma$ and set




def $sl_1(\gamma,\beta) := \deg \Phi_\beta$

$$= \int_{S^2} \Phi_\beta^*\omega \in \mathbb{Z}$$

on board

* $sl_1(\bigcirc, \bigcirc) = 1$

* in general, $sl_1(\gamma,\beta)$ is an odd integer.

* As a function of $\gamma$ alone, it is defined up an even integer.

$$\eta(\gamma) := \frac{1}{4\pi}\int_{C_2(S^1)} \Phi^*\omega$$

$C_2(S^1) \leftarrow$ not compact!

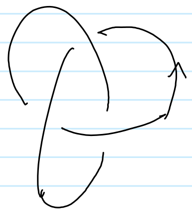
instead consider $sl_2(\gamma,\nu) := l(\gamma, \gamma^{+\nu})$

$\gamma^{+\nu}(s) := \gamma(s) + \epsilon\nu(s)$ . . . framing, framed knots,

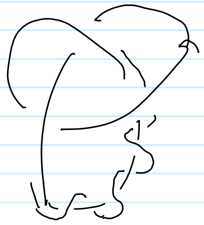
framings $\sim \mathbb{Z}$ as an affine set!

(compare two framings by a map $S^1 \to S^1$)

$\longrightarrow sl_2 = 0$



in BB framing, $sl_2 = 3$



0-framing.

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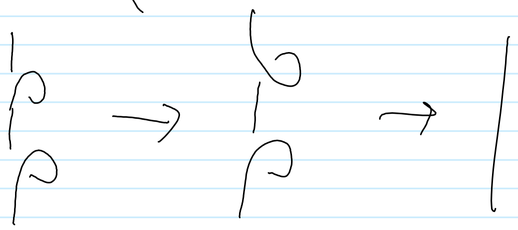
Aside: (Framed immersions $S^1 \to \mathbb{R}^3$) $\Leftrightarrow SO(3)$



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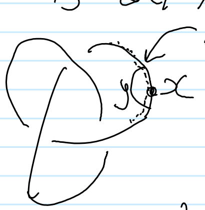
Overall, $sl_2$ is an integer, and as a function of $\gamma$ alone, it is defined mod $\mathbb{Z}$.

$\eta(\gamma) - sl_2(\gamma, \nu) =$  $= \lim_{\delta \to 0} \int_{S^1_x} \int_{b_y^{x,\delta}} \left(\text{pull back of } \omega\right) =$

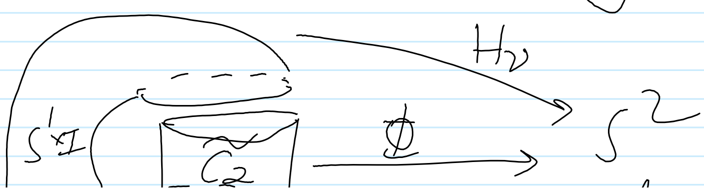
$= \int_{S^1_x} \left( \begin{array}{l} \text{a bdd quantity } \lambda \\ \text{computable from } \gamma, \nu \text{ near } x. \end{array} \right)$

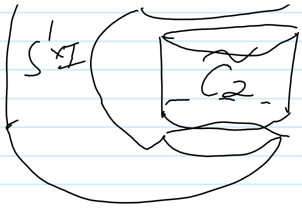
1. if $\nu = \frac{\ddot{\gamma}}{\|\ \|}$ "the normal of $\gamma$" then $\lambda$ is the Frenet–Serret torsion $\tau$: with $n = \ddot{\gamma}/\|\ \|$, $\tau = \dot{n} \cdot (\dot{\gamma} \times n)$

done line

2. Otherwise, $\lambda$ is "the drift rel. to the Riemanian connection".
3. An alternative definition of $sl_2$:

a $\nu$ defines a homotopy $H_\nu : \dot{\gamma} \to -\dot{\gamma}$,






$\xrightarrow{\Phi} S^2$

overall get $\Phi_\nu : T^2 \to S^2$
& set $sl_2 = \deg \Phi_\nu$

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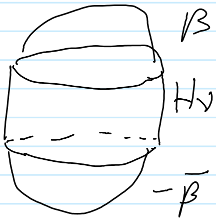
There is a pairing $\langle \nu, \beta \rangle \in \mathbb{Z}$ between framings and swaddling maps:




$\xrightarrow{\Psi_{\nu,\beta}} S^2 \qquad \langle \nu, \beta \rangle = \deg \Psi_{\nu,\beta}$

<u>Thm</u> By declaring $\beta \leftrightarrow \nu \Leftrightarrow \langle \nu, \beta \rangle = 0$, there is a bijection between (homotopy classes of) swaddling maps and odd framings. If $\beta \leftrightarrow \nu$, then $sl_1(\gamma, \nu) = sl_2(\gamma, \beta)$.

<u>Proof</u> HW.