Friday-7 AKT on 140228: Scratch

February-26-14 9:27 AM

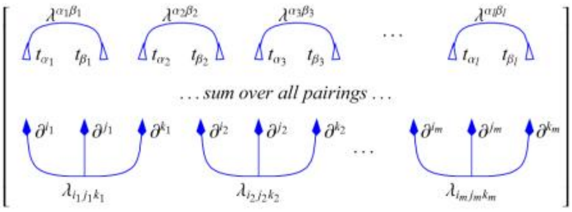
From Gaussian Integration to Feynman Diagrams

We wish to understand $\displaystyle\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\,\mathrm{hol}_\gamma(A)\exp\left(\frac{ik}{4\pi}\int_{\mathbb{R}^3}\mathrm{tr}A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right).$

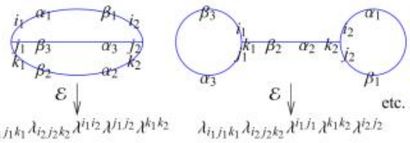
As a warm up, suppose (λ_{ij}) is a symmetic positive definite matrix and (λ^{ij}) is its inverse. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C:=\frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\lambda_{ij}x^ix^j+\frac{1}{6}\lambda_{ijk}x^ix^jx^k} = Ce^{\frac{1}{6}\lambda_{ijk}\partial^i\partial^j\partial^k}e^{\frac{1}{2}\lambda^{\alpha\beta}t_\alpha t_\beta}\Big|_{t_\alpha=0} = \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C}{6^m m!2^l l!}\left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m\left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$=\sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C}{6^m m!2^l l!}$$



$$=\sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C}{6^m m!2^l l!}\sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman diagrams } D}}\mathcal{E}(D)$$

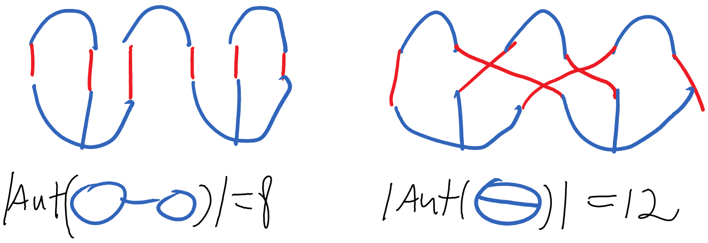


$$=C\sum_{\substack{\text{unmarked Feynman}\\ \text{diagrams } D}}\frac{\mathcal{E}(D)}{|\mathrm{Aut}(D)|}.$$

Claim. The number of pairings that produce a given unmarked Feynman diagram D is $\frac{6^m m!2^l l!}{|\mathrm{Aut}(D)|}$.

Proof of the Claim. The group $\underbrace{[(S_3)^m\rtimes S_m]\times[(S_2)^l\rtimes S_l]}_{6^m m!}$ acts on the set of pairings, the action is transitive on the set of pairings P that produce a given D, and the stabilizer of any given P is $\mathrm{Aut}(D)$. □

Examples:



The Fourier Transform:

$(F:V\to\mathbb{C})\Rightarrow(\tilde{F}:V^*\to\mathbb{C})$

via $\tilde{F}(\varphi)=\int_V F(v)e^{-i\langle\varphi,v\rangle}dv.$

Simple Facts:

1. $\tilde{F}(0)=\int_V F(v)dv.$
2. $\frac{\partial}{\partial\varphi_i}\tilde{F}\sim\widetilde{v^iF}.$
3. $\widetilde{(e^{Q/2})}\sim e^{-Q^{-1}/2}$ where $Q^{-1}(\varphi):=\langle\varphi,L^{-1}\varphi\rangle$ (that's the heart of the Fourier Inversion Formula).

V: Vector space

dV: Lebesgue's measure on V.

Q: A quadratic form on V; $Q(v)=\langle Lv,v\rangle$ where $L:V\to V^*$ is linear

Compute $I=\int_V dv\,e^{\frac{1}{2}Q+P}$

$$\text{"="}\sum_{m=0}^\infty\frac{1}{m!}\int_V dv\,P^m e^{Q/2}$$

$$\sim\sum_{m=0}^\infty\frac{1}{m!}P^m(\partial_\varphi)e^{-\frac{1}{2}Q^{-1}(\varphi)}\Big|_{\varphi=0}$$

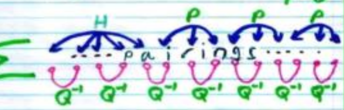
$\pi=2=i=1$

$$=\sum_{m,n=0}^\infty\frac{(-1)^n}{2^n m!n!}P^m(\partial)(Q^{-1})^n\Big|_{\varphi=0}$$

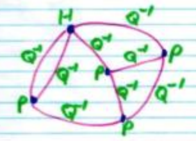
So $\int_V H(v)e^{\frac{1}{2}Q+P}dv$

$$\sim H(\partial)e^{P(\partial)}e^{-Q^{-1}(\varphi)/2}\Big|_{\varphi=0}$$

is

$\sum$ 

$$=\sum_{\text{Diagrams}}c(D)\begin{pmatrix}\text{Products of}\\ Q^{-1}\text{'s, }P\text{'s}\\ \text{and one }H\end{pmatrix}$$



Physics: "Physics should be gauge invariant"

Math: Could cure $\int e^{i\int A\wedge dA} \int_{\gamma_1} A \int_{\gamma_2} A$ thanks to "invariance".

קשרים ודיאגרמות פיינמן, 14 לינואר 2002

$V \longrightarrow V\times M \downarrow M$

קשר/מגזן: $A\in \Omega^1(M, \mathrm{End}\,V)$

אם $s: M \to V$ חתך, אז

$D_A s = ds + As$; $(D_A s)(\xi) = \xi s + A(\xi)(s)$

הרחבת כיול: $g: M \to \mathrm{Aut}(V)$; $D \mapsto D^g = g^{-1} D g$

$A \mapsto g^{-1}dg + g^{-1}Ag$

הולונומיה/העתקה לאורך: $\gamma:[0,1]\to M$, $v_0 \in V$

מחפשים $\tilde\gamma:[0,1]\to V$ כך ש

$(\gamma^*(D_A))(\tilde\gamma) = 0 \Rightarrow \left(\frac{d}{dt} + A(\dot\gamma(t))\right)\tilde\gamma = 0$

אם A אבלית — $\tilde\gamma(t) = e^{-\int_0^t ds\, A(\dot\gamma(s))} v_0$; אחרת

$\tilde\gamma(t) = \underbrace{\left(I - \int_0^t ds_1 A(\dot\gamma(s_1)) + \cdots\right)}_{hol_\gamma(A)(t)} v_0$ $\qquad \frac{d}{dt} hol_\gamma(A)(t) = -A(\dot\gamma)\, hol_\gamma(A)(t)$

טענה $hol_\gamma(A^g) = g(\gamma(1))^{-1} hol_\gamma(A)\, g(\gamma(0))$ הוכחה ...

מסקנה אם γ סגורה אז $tr(hol_\gamma(A))$ שמור-כיול.

הגדרה תורת chern-simons:

$CS(A) = \int_M tr(A\wedge dA) + \frac{2}{3} A\wedge A\wedge A$

טענה גם $CS(A)$ שמורת כיול.