Friday-7 AKT on 140228: Feynman Diagrams, Connections

February-27-14 8:38 PM

HW meeting & class chat at 6PM!

Dror Bar-Natan: Classes: 1314: AKT-14: http://drorbn.net/index.php?title=AKT-14

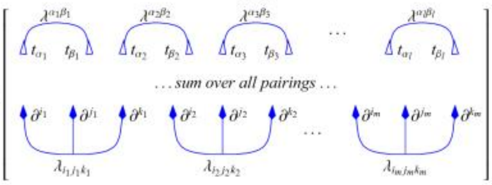
From Gaussian Integration to Feynman Diagrams

We wish to understand $\displaystyle\int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A\,\mathrm{hol}_\gamma(A)\exp\left(\frac{ik}{4\pi}\int_{\mathbb{R}^3}\operatorname{tr}A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right).$

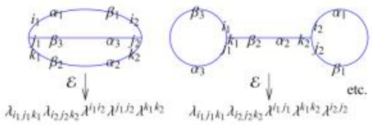
As a warm up, suppose (λ_{ij}) is a symmetric positive definite matrix and (λ^{ij}) is its inverse. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

$$\int_{\mathbb{R}^n} e^{-\frac12\lambda_{ij}x^ix^j+\frac16\lambda_{ijk}x^ix^jx^k} = Ce^{\frac16\lambda_{ijk}\partial^i\partial^j\partial^k}e^{\frac12\lambda^{\alpha\beta}t_\alpha t_\beta}\Big|_{t_\alpha=0} = \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C}{6^m m!2^l l!}\left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m\left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C}{6^m m!2^l l!}$$



$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C}{6^m m!2^l l!}\sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman diagrams } D}}\mathcal{E}(D)$$

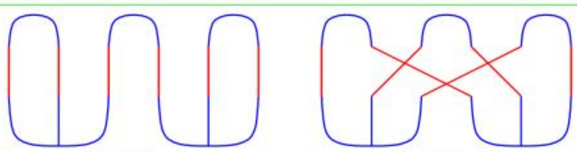


$$= C\sum_{\substack{\text{unmarked Feynman}\\ \text{diagrams } D}}\frac{\mathcal{E}(D)}{|\mathrm{Aut}(D)|}.$$

Claim. The number of pairings that produce a given unmarked Feynman diagram D is $\frac{6^m m!2^l l!}{|\mathrm{Aut}(D)|}$.

Proof of the Claim. The group $G_{m,l} := [(S_3)^m \rtimes S_m]\times[(S_2)^l \rtimes S_l]$ acts on the set of pairings, the action is transitive on the set of pairings P that produce a given D, and the stabilizer of any given P is $\mathrm{Aut}(D)$. □

Examples.



$|\mathrm{Aut}(D)| = 8$ $\quad$ $|\mathrm{Aut}(D)| = 12$

The Fourier Transform:

$(F: V\to\mathbb{C}) \Rightarrow (\tilde F: V^*\to\mathbb{C})$

via $\tilde F(\varphi) = \int_V F(v)e^{-i\langle\varphi,v\rangle}dv.$

Simple Facts:

1. $\tilde F(0) = \int_V F(v)dv.$
2. $\frac{\partial}{\partial\varphi_i}\tilde F \sim \widetilde{v^iF}.$
3. $\widetilde{\left(e^{Q/2}\right)} \sim e^{-Q^{-1}/2}$

 where $Q^{-1}(\varphi) := \langle\varphi, L^{-1}\varphi\rangle$

 (that's the heart of the Fourier Inversion Formula).

V: vector space

dv: Lebesgue's measure on V.

Q: A quadratic form on V; $Q(v)=\langle Lv,v\rangle$ where $L: V\to V^*$ is linear

Compute $I = \int_V dv\, e^{\frac12 Q+P}$

"=" $\displaystyle\sum_{m=0}^\infty \frac{1}{m!}\int_V dv\, P^m e^{Q/2}$

$\displaystyle\sim\sum_{m=0}^\infty\frac{1}{m!}P^m(\partial_\varphi)e^{-\frac12 Q^{-1}(\varphi)}\Big|_{\varphi=0}$

$\pi=2=i=1$

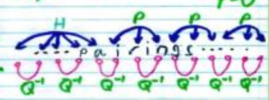
$\displaystyle=\sum_{m,n=0}^\infty\frac{(-1)^n}{2^n m!n!}P^m(\partial)(Q^{-1})^n\Big|_{\varphi=0}$

So $\int_V H(v)e^{\frac12 Q+P}dv$

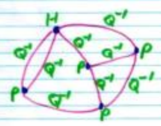
$\sim H(\partial)e^{P(\partial)}e^{-Q^{-1}(\varphi)/2}\Big|_{\varphi=0}$

is

$\sum$



$= \displaystyle\sum_{\text{Diagrams}} c(D)\begin{pmatrix}\text{Products of}\\ Q^{-1}\text{'s, } P\text{'s}\\ \text{and one } H\end{pmatrix}$



Gauge Theory in the simplest case. $V \longrightarrow V\times M$

Let V be a f.d. V.S., $G=\mathrm{Aut}(V)=GL_n$ $\quad$ $\mathfrak{g}=\mathrm{End}(V)=M_{n\times n}$

$g \in \widetilde{\mathcal{G}} = C^\infty(M, G)$ acts on $\Omega^1(M, \mathfrak{g})$ by

$$A \longmapsto A^g = g^{-1}Ag + g^{-1}dg$$

If $s : M \to V$ $\quad D_A = ds + As$ $\qquad D \mapsto D^g = g^{-1}Dg$

for any $D : \Omega^0(M,V) \to \Omega^1(M,V)$

$$D_A \mapsto D_A^g = D_{g^{-1}Ag + g^{-1}dg}$$

done

line

---

physics: "Physics should be gauge invariant"

Math: Could care $\int e^{i\int A \wedge dA} \int_{\gamma_1} A \int_{\gamma_2} A$ thanks

---

Holonomies: Given $\gamma : [0,1] \to M$, $v_0 \in V$, seek

$\tilde{\gamma} : [0,1] \to V$ s.t.

$$\gamma^*(D_A)\tilde{\gamma} = 0 \implies \left(\frac{d}{dt} + A(\dot{\gamma}(t))\right)\tilde{\gamma} = 0$$

If A is Abelian,

$$\tilde{\gamma}(t) = e^{-\int_0^t ds\, A(\dot{\gamma}(s))} v_0$$

Otherwise,

$$\tilde{\gamma}(t) = \underbrace{\left(I - \int_0^t ds_1\, A(\dot{\gamma}(s_1)) + \int_{0 \le s_1 \le s_2 \le t} A(\dot{\gamma}(s_2))A(\dot{\gamma}(s_1)) - \dots\right)}_{hol_\gamma(A)(t)} v_0$$

$$\frac{d}{dt} hol_\gamma(A)(t) = -A(\dot{\gamma}(t))\, hol_\gamma(A)(t)$$

Claim $\quad hol_\gamma(A^g) = g(\gamma(t))^{-1}\, hol_\gamma(A)\, g(\gamma(0))$

Corollary $\quad$ If γ is closed, $tr(hol_\gamma(A))$ is gauge invariant.

Def $\quad CS(A) = \int_M tr(A \wedge dA) + \frac{2}{3} A \wedge A \wedge A$

for 3-D M.

Exercise $\quad CS(A^g) = CS(A)$