FeynmanDiagramsV2 Handout on 140311

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Gaussian Integration, Determinants, Feynman Diagrams

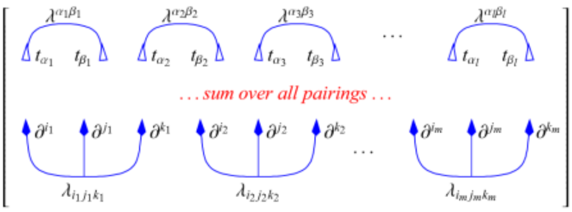
Gaussian Integration. (λ_{ij}) is a symmetic positive definite matrix and (λ^{ij}) is its inverse, and (λ_{ijk}) are the coefficients of some cubic form. Denote by $(x^i)_{i=1}^n$ the coordinates of $\mathbb{R}^n$, let $(t_i)_{i=1}^n$ be a set of "dual" variables, and let ∂^i denote $\frac{\partial}{\partial t_i}$. Also let $C := \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}$. Then

$$\int_{\mathbb{R}^n} e^{-\frac12\lambda_{ij}x^ix^j+\frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k} = \sum_{m\geq 0}\frac{\epsilon^m}{6^m m!}\int_{\mathbb{R}^n}(\lambda_{ijk}x^ix^jx^k)^m e^{-\frac12\lambda_{ij}x^ix^j}$$

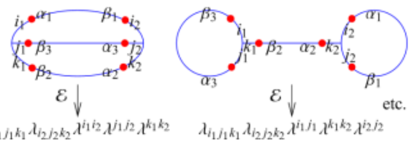
Feynman

$$= \sum_{m\geq 0}\frac{C\epsilon^m}{6^m m!}(\lambda_{ijk}\partial^i\partial^j\partial^k)^m e^{\frac12\lambda^{\alpha\beta}t_\alpha t_\beta}\Big|_{t_\alpha=0} = \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\left(\lambda_{ijk}\partial^i\partial^j\partial^k\right)^m\left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}$$



$$= \sum_{\substack{m,l\geq 0\\ 3m=2l}}\frac{C\epsilon^m}{6^m m!2^l l!}\sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman diagrams } D}}\mathcal{E}(D)$$



$$= C\sum_{\substack{\text{unmarked Feynman}\\ \text{diagrams } D}}\frac{\epsilon^{m(D)}\mathcal{E}(D)}{|\mathrm{Aut}(D)|}.$$

Claim. The number of pairings that produce a given unmarked Feynman diagram D is $\frac{6^m m!2^l l!}{|\mathrm{Aut}(D)|}$.

Proof of the Claim. The group $G_{m,l} := [(S_3)^m \rtimes S_m]\times[(S_2)^l\rtimes S_l]$ acts on the set of pairings, the action is transitive on the set of pairings P that produce a given D, and the stabilizer of any given P is $\mathrm{Aut}(D)$. □

Determinants. Now suppose Q and P_i $(1\leq i\leq n)$ are $d\times d$ matrices and Q is invertible. Then $\int_{\mathbb{R}^n} e^{-\frac12\lambda_{ij}x^ix^j+\frac{\epsilon}{6}\lambda_{ijk}x^ix^jx^k}\det(Q+\epsilon x^iP_i)$

$$= \sum_{m,k\geq 0,\,\sigma\in S_k}\frac{\epsilon^{m+k}}{6^m m!k!}\int_{\mathbb{R}^n}(\lambda_{ijk}x^ix^jx^k)^m\operatorname{tr}\left(\sigma(x^iQ^{-1}P_i)^{\otimes k}\right)e^{-\frac12\lambda_{ij}x^ix^j}$$

= a diagrammatic description

$= \sum$ unmarked diagrams.

A box on Berezin integration

$\boxed{\text{Berezin}}$

A box on interior multiplication.

The Fourier Transform.

$$(F\colon V\to\mathbb{C})\Rightarrow(\tilde f\colon V^*\to\mathbb{C})$$

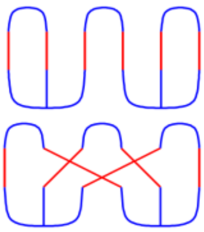
via $\tilde F(\varphi) := \int_V f(v)e^{-i\langle\varphi,v\rangle}dv$. Some facts:

- $\tilde f(0) = \int_V f(v)dv$.
- $\frac{\partial}{\partial\varphi_i}\tilde f \sim \widetilde{v^i f}$.
- $\widetilde{(e^{Q/2})} \sim e^{Q^{-1}/2}$, where Q is quadratic, $Q(v) = \langle Lv, v\rangle$ for $L\colon V\to V^*$, and $Q^{-1}(\varphi) := \langle\varphi, L^{-1}\varphi\rangle$. (This is the key point in the proof of the Fourier inversion formula!)

Examples.

$|\mathrm{Aut}(D)| = 8$

$|\mathrm{Aut}(D)| = 12$



Perturbing Determinants. If Q and P are matrices and Q is invertible,

$$\det(Q+\epsilon P) = \det(Q)\det(I+\epsilon Q^{-1}P)$$
$$= \det(Q)\sum_{k\geq 0}\epsilon^k\operatorname{tr}\left(\bigwedge^k Q^{-1}P\right)$$
$$= \det(Q)\sum_{k\geq 0,\,\sigma\in S_k}\frac{\epsilon^k}{k!}\operatorname{tr}\left(\sigma(Q^{-1}P)^{\otimes k}\right).$$

= a diagrammatic description