Definition A "tensor product" $M \otimes N$ is a module $M \otimes N$ along with a bilinear $\iota : M \times N \longrightarrow M \otimes N$ s.t.

$$\begin{tikzcd} M \times N \arrow[r, "\iota"', "\text{bilinear}"] \arrow[dr, "\rho"', "\text{bilinear}"] & M \otimes N \arrow[d, dashed, "\exists!\ \text{linear}"] \\ & P \end{tikzcd}$$

Thm $M \otimes N$ exists ✓ & is unique up to isomorphism.
today

Example. $\dim V \otimes W = (\dim V)(\dim W)$

Proof of uniqueness.

Example. If $q \in \gcd(a,b)$, $\frac{R}{\langle a \rangle} \otimes \frac{R}{\langle b \rangle} \cong \frac{R}{\langle q \rangle}$
$q = sa + tb$

pf. $[r_1]_a \otimes [r_2]_b \longrightarrow [r_1 \cdot r_2]_q$ $\qquad$ $[q] \otimes [1] = [sa+tb] \otimes [1] = 0$

$[r]_q \longrightarrow [r]_a \otimes [1]_b$ $\qquad$ $[r_1 r_2] \otimes [1] = [r_1][r_2]$

example. $\iota : F(X) \otimes F(Y) \to F(X \times Y)$

1. Always injective! [not so easy!]
2. Isomorphism if X or Y are finite. Matt

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3. Not surjective if $R = \mathbb{Z}$, X, Y are infinite. no do.
[not at all obvious!]

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theorem. $(R\text{-mod}, \oplus, \otimes, 0, R)$ is a "ring". Jacob.

theorem. $(M, N) \longmapsto M \otimes N$ is a "bifunctor".

Example. $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}^n \cong \mathbb{Q}^n$ "Extension of scalars".
$\nwarrow$ a $\mathbb{Q}$-module!

In general, given $\phi: R\to S$ a ring morphism, S is an R module & set $M_S := S\otimes_R M$. Then M_S is an S-module and $R^n_S = S^n$.

Prop. For any domain R there is a unique field $Q(R)$ s.t.

$$\begin{tikzcd} R \arrow[r, "1-1"] \arrow[dr] & Q(R) \arrow[d, dashed, "\exists!"] \\ & F \end{tikzcd}$$

"the field of fractions"

proof: later.

<u>Claim</u> If M is torsion $\left[\begin{matrix}\forall m\in M\ \exists r\in R\setminus\{0\}\\ rm=0\end{matrix}\right]$ Then $M_{Q(R)}=0$.

$a\otimes m = r(\frac{a}{r}\otimes m) = \frac{a}{r}\otimes rm = 0$

<u>Prop</u> If $M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle$, then

Pro

1. $\dim_{Q(R)} M_{Q(R)} = k$
2. $\dim_{R/\langle p\rangle} M_{R/\langle p\rangle} = k + |\{i : p_i \sim p\}|$
3. $\dim_{R/\langle p\rangle} \mathrm{im}(m\mapsto p^s m)_{R/\langle p\rangle} = k + |\{i : p_i\sim p \ \&\ s<s_i\}|$

$R/\langle p\rangle$ is a field because in a PID $\langle p\rangle$ is maximal

as $\mathrm{im}(m\mapsto p^s m) \cong \begin{cases} p^sR\cong R & \text{on } R\\ R/\langle q^t\rangle & \text{on } R/\langle q^t\rangle \quad q\not\sim p\\ 0 & \text{on } R/\langle p^t\rangle \quad s\geq t\\ R/\langle p^{t-s}\rangle & \text{on } R/\langle p^t\rangle \quad s<t\end{cases}$

and $\ker(m\mapsto p^s m)\cong \begin{cases} 0 & \text{on } R\\ 0 & \text{on } R/\langle q^t\rangle \quad q\not\sim p\\ R/\langle p^t\rangle & \text{on } R/\langle p^t\rangle \quad s\geq t\\ R/\langle p^s\rangle & \text{on } R/\langle p^t\rangle \quad s<t\end{cases}$

$R/\langle p^s\rangle \mapsto \ker$ by $[r]_{p^s}\mapsto [p^{t-s}r]_{p^t}$

So such a decomposition is unique!

Localization & Fields of fractions. Let R be a commutative domain

<u>Def</u> A multiplicative subset S of $R\setminus\{0\}$. (contains 1, closed under $\times$)

<u>Examples</u> $R\setminus\{0\}$, $R\setminus P$ (P prime), Powers of $a\neq 0$.

<u>Definition</u> $S^{-1}R = \left\{\frac{r}{s}\right\} \Big/ \frac{r_1}{s_1}\sim\frac{r_2}{s_2}$ if $r_1 s_2 = r_2 s_1$

$\left[\frac{r_1}{s_1}\sim\frac{r_2}{s_2}, \frac{r_2}{s_2}\sim\frac{r_3}{s_3} \Rightarrow r_1 s_2 = r_2 s_1, r_2 s_3 = r_3 s_2 \Rightarrow\right.$

$\left. r_1 s_2 s_3 = r_2 s_1 s_3 = s_1 r_3 s_2 \Rightarrow r_1 s_3 = r_3 s_1 \right]$

$\frac{r_1}{s_1}+\frac{r_2}{s_2}=\dots$

$\frac{r_1}{s_1}\cdot\frac{r_2}{s_2}=\dots$

$R\setminus\{0\}$ — "Field of fractions $Q(R)$"

$R\setminus P$ — "localization at P"

$\{2^n\}$ — "dyadic rationals".

$R\to S^{-1}R$ is injective