Stonehenge

From Stonehenge to Witten Skipping all the Details

Oporto Meeting on Geometry, Topology and Physics, July 2004

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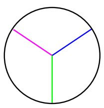
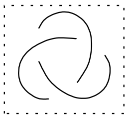
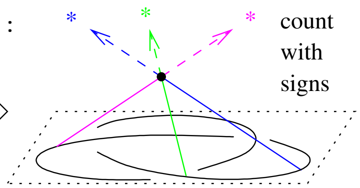
Witten

It is well known that when the Sun rises on midsummer's morning over the "Heel Stone" at Stonehenge, its first rays shine right through the open arms of the horseshoe arrangement. Thus astrological lineups, one of the pillars of modern thought, are much older than the famed Gaussian linking number of two knots.

Recall that the latter is itself an astrological construct: one of the standard ways to compute the Gaussian linking number is to place the two knots in space and then count (with signs) the number of shade points cast on one of the knots by the other knot, with the only lighting coming from some fixed distant star.

$$\langle D,K\rangle_{\text{Stonehenge}} := \left(\begin{array}{l}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{array}\right):$$

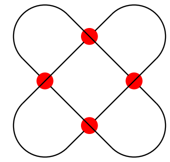
$D=$  $K=$  $\Longrightarrow$ 

count with signs

The Gaussian linking number

$lk($  $) = \frac{1}{2}\sum_{\text{vertical chopsticks}}(\text{signs})$

Carl Friedrich Gauss



$lk=2$

Thus we consider the generating function of all stellar coincidences:

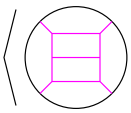
Dylan Thurston

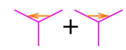
$$Z(K) := \lim_{N\to\infty}\sum_{3\text{-valent } D}\frac{1}{2^c c!\binom{N}{e}}\langle D,K\rangle_{\text{Stonehenge}} D\cdot\left(\begin{array}{c}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{array}\right)\in\mathcal{A}(\circlearrowleft)$$

N := # of stars

c := # of chopsticks

e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:=Span $\left\langle\right.$  $\left.\right\rangle$ / oriented vertices

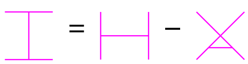
AS:  =0

& more relations

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,



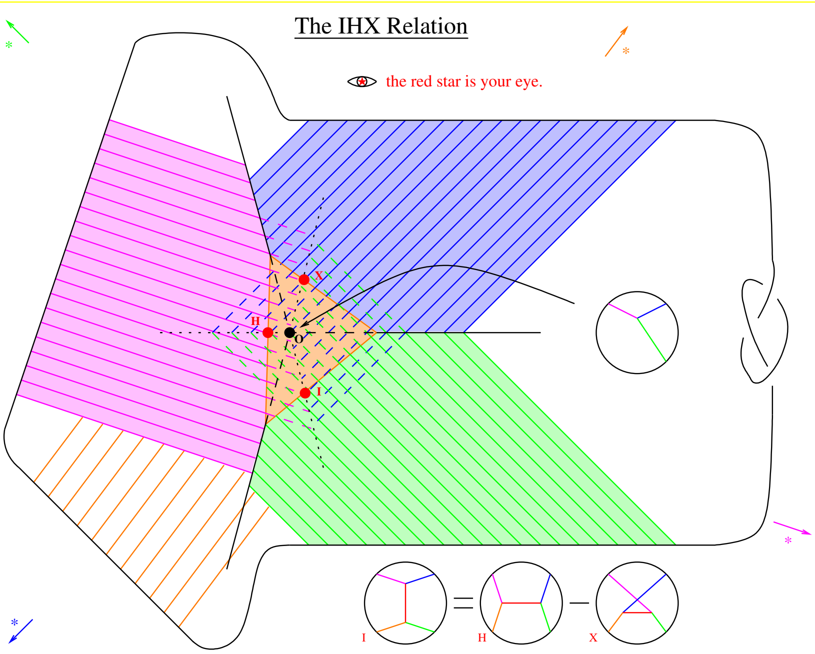
(see below)

An intersection line cuts through the knot – Solution: Impose STU,

(similar argument)

The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.

(not shown here)

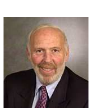


It all is perturbative Chern–Simons–Witten theory:

$$\int_{\mathfrak{g}\text{-connections}}\mathcal{D}A\, hol_K(A)\exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3}\operatorname{tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)\right]$$

$$\to\sum_{D:\text{ Feynman diagram}} W_{\mathfrak{g}}(D)\sum\!\!\!\!\!\!\int\mathcal{E}(D)\to\sum_{D:\text{ Feynman diagram}} D\sum\!\!\!\!\!\!\int\mathcal{E}(D)$$

Shiing–shen Chern



James H Simons

V: vector space

dV: Lebesgue's measure on V.

Q: A quadratic form on V; $Q(V)=\langle LV,V\rangle$ where $L:V\to V^*$ is linear

Compute $I=\int_V dv\, e^{\frac{1}{2}Q+P}$

$$\text{“=”}\sum_{m=0}^{\infty}\frac{1}{m!}\int_V dv\, P^m e^{Q/2}$$

$$\sim\sum_{m=0}^{\infty}\frac{1}{m!}P^m(\partial_\varphi)e^{-\frac{1}{2}Q^{-1}(\varphi)}\Big|_{\varphi=0}$$

$\pi=2=i=1$

$$=\sum_{m,n=0}^{\infty}\frac{(-1)^n}{2^n m!n!}P^m(\partial)(Q^{-1})^n\Big|_{\varphi=0}$$

The Fourier Transform:

$(F:V\to\mathbb{C})\Rightarrow(\tilde{F}:V^*\to\mathbb{C})$

via $\tilde{F}(\varphi)=\int_V F(v)e^{-i\langle\varphi,v\rangle}dv$.

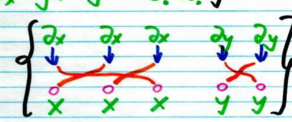
Simple Facts:

1. $\tilde{F}(0)=\int_V F(v)dv$.
2. $\frac{\partial}{\partial\varphi_i}\tilde{F}\sim\widetilde{V^iF}$.
3. $\widetilde{(e^{Q/2})}\sim e^{-Q^{-1}/2}$ where $Q^{-1}(\varphi):=\langle\varphi,L^{-1}\varphi\rangle$

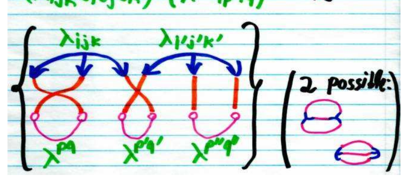
(that's the heart of the Fourier Inversion Formula).

Differentiation and Pairings:

$\partial_x^3\partial_y^2x^3y^2=3!\cdot2!$; indeed,

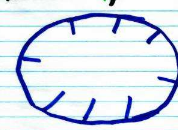


$(\lambda_{ijk}\partial_i\partial_j\partial_k)^2(\lambda^{pq}\varphi_p\varphi_q)^3$ is



In our case,

* Q is d, so Q^{-1} is an integral operator.
* P is $\frac{2}{3}A\wedge A\wedge A$
* H is the holonomy, itself a sum of integrals along the knot K,

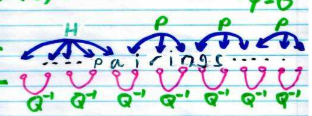


& when the dust settles, we get $Z(K)$!

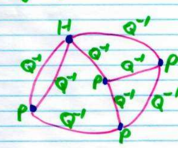
So $\int_V H(v)e^{\frac{1}{2}Q+P}dv$

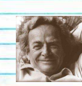
$$\sim H(\partial)e^{P(\partial)}e^{-Q^{-1}(\varphi)/2}\Big|_{\varphi=0}$$

is

$\sum$ 

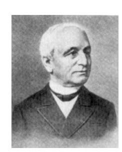
$$=\sum_{\text{Diagrams}}c(D)\left(\begin{array}{c}\text{Products of}\\ Q^{-1}\text{'s, }P\text{'s}\\ \text{and one }H\end{array}\right)$$





Richard Feynman

"God created the knots, all else in topology is the work of man."



Leopold Kronecker

(modified)

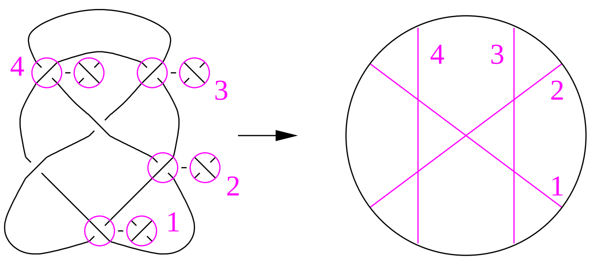
This handout is at http://www.math.toronto.edu/~drorbn/Talks/Oporto–0407

Knotted Trivalent Graphs, Tetrahedra and Associators

HUJI Topology and Geometry Seminar, November 16, 2000

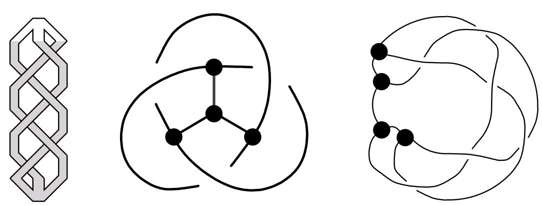
Dror Bar−Natan

Goal: Z:{knots}->{chord diagrams}/4T so that

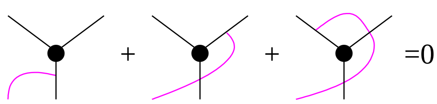


The Miller Institute knot

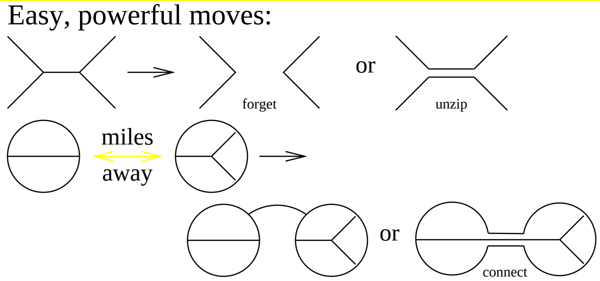
Extend to Knotted Trivalent Graphs (KTG's):

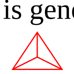


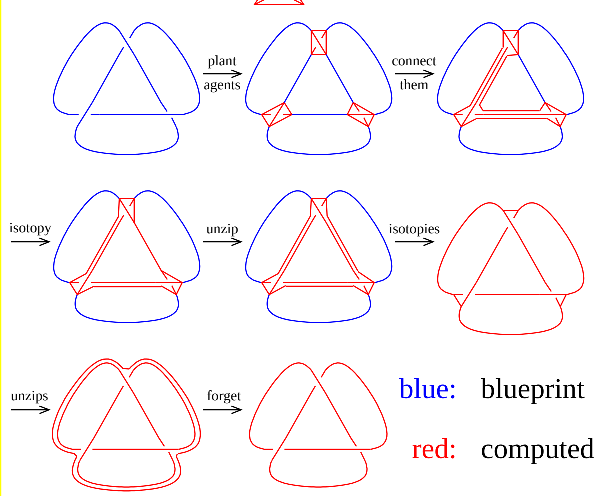
Need a new relation:



Easy, powerful moves:

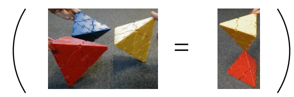


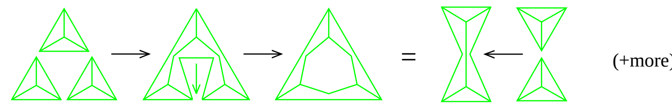
Using moves, KTG is generated by ribbon twists and the tetrahedron  :



blue: blueprint

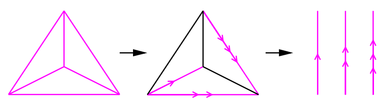
red: computed

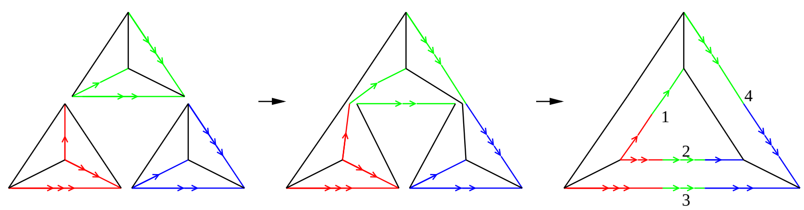
Modulo the relation(s): 

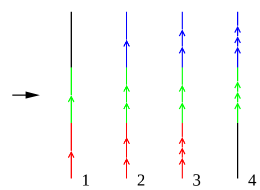


Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi Hopf algebras.

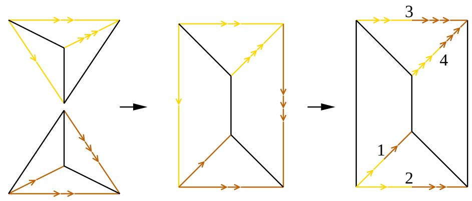
Proof.

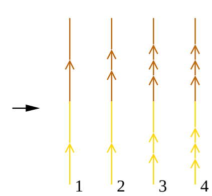
 $:= \Phi \in \mathcal{A}(\uparrow_3)$





$= (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (1 \otimes \Phi) \in \mathcal{A}(\uparrow_4)$





$= (\Delta \otimes 1 \otimes 1)(\Phi) \cdot (1 \otimes 1 \otimes \Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$

Further directions:

1. Relations with perturbative Chern−Simons theory.
2. Relations with the theory of 6j symbols
3. Relations with the Turaev−Viro invariants.
4. Can this be used to prove the Witten asymptotics conjecture?
5. Does this extend/improve Drinfel'd's theory of associators?

This handout is at http://www.ma.huji.ac.il/~drorbn/Talks/HUJI−001116