October 27, hour 21: Rings, ideals, isomorphism theorems, prime and maximal ideals

October-25-11
11:22 AM

Read Along. Selick 2.1-23 Term test. Discussion at 10:45 also return HW2.

Goal. 1. Rings, ideals, isomorphisms.

2. Prime & maximal ideals, domains and fields.

Definition 2.1.1. *A **ring** consists of a set R together with binary operations $+$ and $\cdot$ satisfying:*

1. $(R,+)$ *forms an abelian group,*
2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\forall a,b,c \in R$,
3. $\exists 1 \neq 0 \in R$ *such that* $a \cdot 1 = 1 \cdot a = a$ $\forall a \in R$, *and*
4. $a \cdot (b+c) = a \cdot b + a \cdot c$ *and* $(a+b) \cdot c = a \cdot c + b \cdot c$ $\forall a,b,c \in R$.

Also define: Commutative ring.

Examples. $\mathbb{Z}$, $R[x]$, $M_{n\times n}(R)$

Morphisms, (Examples: 1. $\mathbb{Z} \longrightarrow \mathbb{Z}/n$ 2. $R \longrightarrow R[x]$ at deg 0 3. $R \to M_{n\times n}(R)$ as diag 4. $ev_u : R[x] \to R$ (if R is commutative) 5. $M_{n\times n}(R[x]) \cong M_{n\times n}(R)[x]$)

Added Dec 2012: perhaps I should have proven Cayley-Hamilton right here:

$\det(tI-A)\cdot I = \operatorname{adj}(tI-A)(tI-A) = (\sum B_i t^i)(tI-A)$; now substitute $t=A$. The B_i's commute with A because $(tI-A)\operatorname{adj}(tI-A) = \operatorname{adj}(tI-A)(tI-A)$.

im, subring, ker, ideal.

done line

Q. Is every ideal a quotient?

Ans. Define R/I.

The Isomorphism theorems. 1. $f: R \to S \Rightarrow R/\ker(f) = \operatorname{im} f$.

2. $\frac{A+I}{I} \cong A/A\cap I$ $A \subset R$ subring, $I \subset R$ ideal.

3. $I \subset J \subset R$ ideals $\Rightarrow \dfrac{R/I}{J/I} \cong R/J$

4. Given an ideal I of R, there's a bijection between ideals $I \subset J \subset R$ & ideals of R/I.