October 25, hours 19-20: Term Test

October-23-11
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See http://katlas.math.toronto.edu/drorbn/index.php?title=11-1100/Term_Test

Subjects. 1. The NCGE story.

2. The isomorphism theorems.

3. Jordan Hölder, Solvable groups.

4. Permutations, simplicity of A_n. ✓

5. G-sets. ✓

6. The Sylow theorems, small examples ✓

7. Semi-direct products, braids. ✓

$(123)(345)=(12345)$

$(123)(234)=(12)(34)$

$(12)(34)(123)=(1)(243)$

$(12)(34)(23)(45)=(12453)$

$(123)^{(345)}\sim(124)$

1. Let n be odd. Prove that a subgroup of S_n which contains both (123) & $(123\ldots n)$ is A_n.

(Hint: Conjugate your way up, do not use NCGE).

2. Prove that the G-sets G/H_1 & G/H_2 are isomorphic iff H_1 is conjugate to H_2.

$H_1 \overset{\phi}{\longmapsto} gH_2$ $\qquad h\in H_1 \mapsto hg\in gH_2$

$gH_2 \longmapsto H_1$ $\qquad g^{-1}hg\in H_2$

$gH_2 \longmapsto H_1$ ying

$H_2 \longmapsto g^{-1}H_1$

3. 1. Prove that the semi-direct product of two torsion-free groups is torsion-free.
   2. Prove that there is no braid β s.t. $\beta^n = e$.
4. Sylow-4. (modeled on last year).

Aside: $S_3/\langle(12)\rangle = \left\{ \begin{array}{l} \{[123], [213]\} \\ \{[132], [312]\} \\ \{[231], [321]\} \end{array} \right\}$

Rough Grading Key:

Solve the following 4 problems. Each problem is worth 25 points. You have an hour and fifty minutes. **Neatness counts! Language counts!**

Problem 1. Let G be a finite group, let p be a prime number, and let α be the largest natural number such that $p^\alpha \mid |G|$.

-3

~~13~~ 14 1. Prove that there is a subgroup P of G whose order is p^α. (You are not allowed to use the Sylow theorems, of course). one case wrong.

(-3) $|x|=p$ 11 2. Suppose that $x \in G$ is an element whose order is a power of p, and suppose that x normalizes P. Show that $x \in P$.

Problem 2. A group G is said to be "torsion free" if every non-trivial element thereof has infinite order.

15 1. Prove that a semi-direct of two torsion free groups is again torsion free. (-6) only deduced $h=e_H$ not $g=e_G$.

10 2. Let β be a pure braid on n strands. Prove that if $\beta^7 = e$ then $\beta = e$.

Problem 3. Let H_1 and H_2 be subgroups of some group G. Prove that the left G-sets G/H_1 and G/H_2 are isomorphic (as left G-sets) iff the subgroups H_1 and H_2 are conjugate.

Problem 4.

12 1. Let G be a subgroup of S_n that contains both the transposition (12) and the n-cycle $(123\ldots n)$. Prove that $G = S_n$. (Hint: Conjugate your way up, do not use non commutative Gaussian elimination).

12 2. Let n be odd and let G be a subgroup of S_n that contains both the 3-cycle (123) and the n-cycle $(123\ldots n)$. Prove that $G = A_n$. (Hint: For the lower bound, conjugate your way up, do not use non commutative Gaussian elimination).

4 3. In the previous part, what if n is even?

28

Good Luck!

Problem 3:

$\Leftarrow$: If $H_2 = g^{-1} H_1 g$

5 define $\Psi: G/H_1 \to G/H_2$ by $\Psi(xH_1) = xgH_2$

2 check well-def: $xh_1H_1 \xrightarrow{\Psi} xh_1gH_2 = xgh_1^gH_2 = xgH_2$

2 check G-set morphism.

2 check injectivity.

2 check surjectivity.

$\Rightarrow$: If $\phi: G/H_1 \to G/H_2$ is an isomorphism,

5 $\phi(H_1) = gH_2$ for some g

4 $gH_2 = \phi(h_1H_1) = h_1gH_2 \Rightarrow g^{-1}h_1g \in H_2 \Rightarrow g^{-1}H_1g \subset H_2$

3 but also $\phi^{-1}(gH_2) = H_1$ so

$$\phi^{-1}(H_2) = g^{-1}H_1$$

so by analogy, $gH_2g^{-1} \subset H_1$

$$\Rightarrow g^{-1}H_1g = H_2$$

$gx = y \Rightarrow x = g^{-1}y$

Further Thoughts

Upon further thought and after talking to some students and some email exchanges, I think I made (at least) three mistakes around this term exam:

- It was too long, overall, especially given my insistence that "neatness counts, language counts". Asking just three of the four questions would have been enough.
- Question 3 required too much abstract thought given the time constraints. I should have either given a significant hint or left it out.
- I shouldn't have "rushed to publish" - I should have given myself a little more time to think before returning the exams. Marking up is always possible, but it is better done before the grades are first published, not after.

Anyway, in light of the first point above, I will consider this exam as if the perfect mark in it was 75, effectively multiplying every grade by a factor of 4/3. The few people whose grade now is more than 100 get to keep those extra points, though the maximal possible grade in this class remains an A+.

People who haven't tried don't realize how hard learning may be, forcing you to confront your fears and insecurities (yet it is well worth it!). Try teaching (recommended!) and you'll see it's hard too. After more than 20 years I still make mistakes.

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