October 18, hours 16-17: Braids, some groups of small order, solvable groups, rings

October-16-11
8:25 PM

Read Along. Selick 1.8, 1.10, 1.11, 2.1.

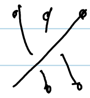
Riddle Along. $\boxed{\forall \alpha \in \mathbb{R}\ \exists a_i \in \mathbb{Q} \text{ s.t. } a_i \to \alpha}$ $\boxed{\mathbb{Q} \cap [0,\alpha]}$ what do these solve?

Term Test. material: everything; sample: See 2010.

Agenda. more semi-directs; tiny bit on solvable groups; rings.

Semi-Direct Products. Given N, H & $\phi: H \xrightarrow{\text{mor}} Aut(N)$,

$$N \rtimes_\phi H := \left(N \times H,\ (n_1,h_1)\cdot(n_2 h_2) = (n_1 \phi_{h_1}(n_2),\ h_1 h_2)\right)$$

Big Example. $B_n = \pi_1\left(\left(\mathbb{C}^n - \{\text{diags}\}\right)/S_n\right) =$ 

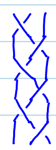
$$B_n = \left\langle \sigma_1, \dots \sigma_{n-1} : \begin{array}{l} \sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle$$

an aside on free groups, generators & relations.

Near class done line

$\pi : B_n \longrightarrow S_n \qquad PB_n = \ker \pi$

$PB_n \lhd B_n$ yet not $B_n = PB_n \rtimes S_n$



3 reasons why I like this one:
1. knotted $20's.
2. Borromean,
3. juggling

$\rho: PB_n \to PB_{n-1} \qquad \ker \rho = F_{n-1}$ and

$$PB_n = F_{n-1} \rtimes PB_{n-1} = F_{n-1} \rtimes \left(F_{n-2} \rtimes \left(\dots (F_2 \rtimes \mathbb{Z}) \dots \right)\right)$$

Groups of order 21. $\mathbb{Z}/21$, $\mathbb{Z}/7 \rtimes \mathbb{Z}/3 = \langle x \rangle \rtimes \langle y \rangle$

$Aut(\mathbb{Z}/7) = \mathbb{Z}/6 = \langle \phi_3 \rangle$; $\phi_3(x) = x^3$; $x^y = x$ or x^2 or x^4 (isomorphic)

(iso: if $x^y = x^2$ & $\bar{y} = y^2$ then $x^{\bar{y}} = x^4$)

Groups of order 12. If $|G| = 12$, $P_4 = \mathbb{Z}/4$ or $(\mathbb{Z}/2)^2$, $P_3 = \mathbb{Z}/3$, and at least one of those is normal, for there's not enough room for 4 P_3 & 3 P_4's. So G is a semi-direct product:

$\mathbb{Z}/4 \rtimes \mathbb{Z}_3$: must be $\mathbb{Z}/4 \times \mathbb{Z}/3 = \mathbb{Z}/12$

$(\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}_3$: Either direct; $\mathbb{Z}/2 \times \mathbb{Z}/6$

or the fun action of $\mathbb{Z}/3$ on $(\mathbb{Z}/2)^2$, giving A_4

$\langle (123) \rangle$ $\qquad$ e, $(12)(34)$, $(13)(24)$, $(14)(23)$

$\mathbb{Z}/3 \rtimes (\mathbb{Z}/2 \times \mathbb{Z}/2)$: Either direct or $D_6 \times \mathbb{Z}/2 = D_{12}$

$\mathbb{Z}/3 \rtimes \mathbb{Z}/4$: Either direct or $\mathbb{Z}/3 \rtimes \mathbb{Z}/4$ $\qquad$ done, but A_4 & D_{12} not well

$\mathbb{Z}/3 \rtimes \mathbb{Z}/4$: Either direct or $\mathbb{Z}/3 \rtimes \mathbb{Z}/4$ done, but A_4 & D_{12} not well done

Solvable Groups. Def G is solvable if all quotients in its Jordan-Hölder series are Abelian.

Thm 1. If $N \trianglelefteq G$, G is solvable iff N & G/N are.

2. If $H < G$ and G is solvable, so is H.

$A \trianglelefteq B$ $H \cap A \trianglelefteq H \cap B$? ✓ $\frac{H \cap B}{H \cap A} \longrightarrow B/A$ by $[b]_{H\cap A} \to [b]_A$ is injective.

Rings.

Definition 2.1.1. *A* ***ring*** *consists of a set R together with binary operations + and · satisfying:*

1. $(R, +)$ *forms an abelian group,*
2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\forall a, b, c \in R$,
3. $\exists 1 \neq 0 \in R$ *such that* $a \cdot 1 = 1 \cdot a = a$ $\forall a \in R$, *and*
4. $a \cdot (b + c) = a \cdot b + a \cdot c$ *and* $(a + b) \cdot c = a \cdot c + b \cdot c$ $\forall a, b, c \in R$.

Also define: Commutative ring.

Examples. $\mathbb{Z}$, $R[x]$, $M_{n\times n}(R)$

Morphisms, (Examples: 1. $\mathbb{Z} \longrightarrow \mathbb{Z}/n$
2. $R \longrightarrow R[x]$ at deg 0
3. $R \to M_{n\times n}(R)$ as diag
4. $ev_u : R[x] \to R$ (if R is commutative)
5. $M_{n\times n}(R[x]) \cong M_{n\times n}(R)[x]$)