

IT 2C3W: $[M \text{ f.g.}/R \text{ PID} \Rightarrow M \cong R^k \oplus \bigoplus R\langle p_i, s_i \rangle]$
 \Rightarrow structure of f.g. Abelian groups, J.C.F.

Riddle Along. Allowig AC but not CH, can you find a chain $(A, B \in \mathcal{S} \Rightarrow (A \subset B) \vee (B \subset A))$ of measure 0 subsets of \mathbb{R} whose union isn't of measure 0?

Today. The "ring" of modules.

Reminder. An R -module: "A vector space over a ring".

Examples. 1. V.S. over a field.

2. Abelian groups over \mathbb{Z} .

3. Given $T: V \rightarrow V$, V over $F[x]$.

4. Given ideal $I \subset R$, R/I over R .

5. Column vectors R^n over $M_{n \times n}$ (left module R -mod)
 row vectors $(R^n)^T$ over $M_{n \times n}$ (right module $\text{mod-}R$)

Def/claim. R -mod & $\text{mod-}R$ are categories.

Def/claim. Submodules, $\ker \phi$, $\text{im } \phi$, M/N

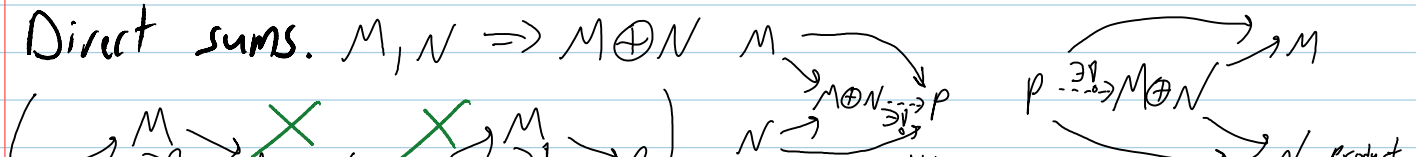
Boring Theorems. 1. $\phi: M \rightarrow N \Rightarrow M/\ker \phi \cong \text{im } \phi$

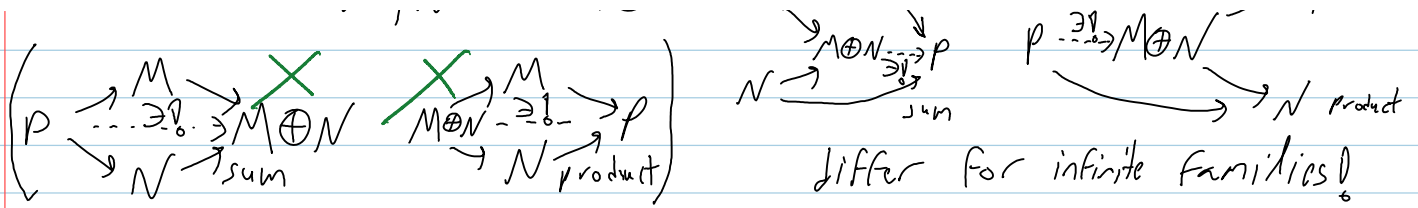
2. $A, B \subset M \Rightarrow (A+B)/B \cong A/(A \cap B)$

3. $A \subset B \subset M \Rightarrow M/A/B/A \cong M/B$

4. Also dual.

Direct sums. $M, N \Rightarrow M \oplus N$





$$\text{Hom}\left(\bigoplus_i N_i, \bigoplus_j M_j\right) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \text{Hom}(M_j, N_i) \right\}$$

done
link

Example: $\dim(V \oplus W) = \dim V + \dim W.$

Example: if $\gcd(a,b)=1$ $1=sa+tb$ [e.g., if R is a PID]

$$\frac{R}{\langle a \rangle} \oplus \frac{R}{\langle b \rangle} \cong \frac{R}{\langle ab \rangle} \quad \text{via} \quad \begin{array}{ccc} R/\langle a \rangle & \xrightarrow{t \cdot b} & R/\langle ab \rangle \\ \oplus & & \\ R/\langle b \rangle & \xrightarrow{s \cdot a} & R/\langle ab \rangle \end{array} \begin{array}{c} \xrightarrow{1} R/\langle a \rangle \\ \oplus \\ \xrightarrow{1} R/\langle b \rangle \end{array}$$

$$\mathbb{Z}/7 \oplus \mathbb{Z}/11 \oplus \mathbb{Z}/13 \cong \mathbb{Z}/77 \oplus \mathbb{Z}/13 \cong \mathbb{Z}/1001 \quad \text{"the chinese remainder theorem"}$$