November 17, hour 28: The ring of modules

November-16-11
8:55 AM

IT2C3W: $[M \text{ f.g.}/R \text{ PID} \Rightarrow M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle]$

$\Rightarrow$ structure of f.g. Abelian groups, J.C.F.

Riddle Along. Allowing AC but not CH, can you find a chain $(A, B \in \mathcal{B} \Rightarrow (A \subset B) \vee (B \subset A))$ of measure 0 subsets of $\mathbb{R}$ whose union isn't of measure 0?

Today. The "ring" of modules.

Reminder. An R-module: "A vector space over a ring".

Examples. 1. V.S. over a field.

2. Abelian groups over $\mathbb{Z}$.

3. Given $T: V \to V$, V over $F[x]$.

4. Given ideal $I \subset R$, R/I over R.

5. Column vectors R^n over $M_{n\times n}$ (Left module R-mod)
row vectors $(R^n)^T$ over $M_{n\times n}$ (right module mod-R)

Def/Claim. R-mod & mod-R are categories.

Def/claim. Submodules, $\ker \phi$, $\operatorname{im} \phi$, M/N

Boring Theorems. 1. $\phi: M \to N \Rightarrow M/\ker\phi \cong \operatorname{im}\phi$

2. $A, B \subset M \Rightarrow (A+B)/B \cong A/(A\cap B)$

3. $A \subset B \subset M \Rightarrow (M/A)/(B/A) \cong M/B$

4. Also dull.

Direct sums. $M, N \Rightarrow M \oplus N$

$$\begin{tikzcd} M \arrow[dr] \arrow[drr, bend left] & & \\ & M\oplus N \arrow[r, dashed, "\exists!"] & P \\ N \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd} \qquad \begin{tikzcd} & & M \\ P \arrow[r, dashed, "\exists!"] \arrow[urr, bend left] \arrow[drr, bend right] & M\oplus N \arrow[ur] \arrow[dr] & \\ & & N \end{tikzcd}$$

Product

$$\left(\begin{tikzcd} & M \arrow[dr] & \\ P \arrow[ur] \arrow[dr] \arrow[rr, dashed, "\exists!"] & & \cancel{M\oplus N} \\ & N \arrow[ur, "\text{sum}"'] & \end{tikzcd} \qquad \begin{tikzcd} & M \arrow[dr] & \\ \cancel{M\oplus N} \arrow[ur] \arrow[dr] \arrow[rr, dashed, "\exists!"] & & P \\ & N \arrow[ur, "\text{product}"'] & \end{tikzcd}\right)$$

$$\begin{tikzcd} M \arrow[r] & M\oplus N \arrow[r, dashed, "\exists!"] & P \\ N \arrow[ur, "\text{sum}"'] & & \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, dashed, "\exists!"] \arrow[dr] & M\oplus N \arrow[d] \\ & N \ \text{product} \end{tikzcd}$$

differ for infinite families!

$$\mathrm{Hom}\left(\bigoplus^{n} N_j, \bigoplus^{m} M_i\right) = \left\{ \begin{pmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn} \end{pmatrix} : a_{ij} \in \mathrm{Hom}(M_i, N_j) \right\}$$

done
link

Example: $\dim(V\oplus W) = \dim V + \dim W$.

Example: if $\gcd(a,b)=1$ $\quad 1 = sa+tb$ $\quad$ [e.g., if R is a PID]

$$\frac{R}{\langle a\rangle} \oplus \frac{R}{\langle b\rangle} \cong \frac{R}{\langle ab\rangle} \quad \text{via} \quad \begin{tikzcd} R/\langle a\rangle \arrow[dr, "t\cdot b"] & & R/\langle a\rangle \\ \oplus & R/\langle ab\rangle \arrow[ur, "1"] \arrow[dr, "1"'] & \oplus \\ R/\langle b\rangle \arrow[ur, "s\cdot a"'] & & R/\langle b\rangle \end{tikzcd}$$

$$\mathbb{Z}/7 \oplus \mathbb{Z}/11 \oplus \mathbb{Z}/13 \cong \mathbb{Z}/77 \oplus \mathbb{Z}/13 \cong \mathbb{Z}/1{,}001$$ "the chinese remainder theorem"