September 28, hours 7-8: The Butterfly and Jordan-Holder

September-21-10
12:42 PM

on board

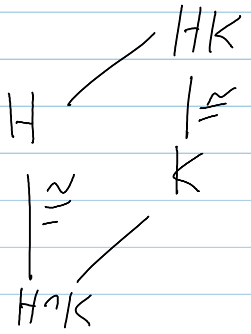
Class photo at break! HW1 at night!

Read Along: Lang's book I-3, Selick notes 1.6, Hungerford's book 7.10.

Review 2nd Iso: $H, K < G$, $H < N_G(K)$

$\Rightarrow H\cap K \triangleleft H$, $K \triangleleft HK$ and

$HK/K = H/H\cap K$.



The Butterfly. $1 < a \triangleleft A < G$, $1 < b \triangleleft B < G$

(Following Lang)

$\Rightarrow a(A\cap b) \triangleleft a(A\cap B)$,

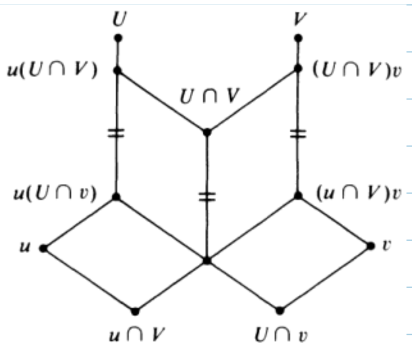
$(a\cap B)b \triangleleft (A\cap B)b$,

and $a(A\cap B)/a(A\cap b) \cong (A\cap B)b/(a\cap B)b$

"The B quotient in the A scale" $\cong$ "The A quotient in the B scale"

(A catchy phrasing is a memory aid, not a proof & not even an intuition)

on board



(likewise for pictures of insects)

(Picture from Lang's book)

Jordan-Hölder Now follows.... Use "normal towers", "refinement", "composition series"

Proof of Butterfly. Normality is obvious.

$(A\cap B)\cdot a(A\cap b) = a(A\cap b)(A\cap B) = a(A\cap B)$, so

$$a(A\cap B)/a(A\cap b) = \overset{H}{\boxed{(A\cap B)}}\overset{K}{\boxed{a(A\cap b)}}/\overset{K}{\boxed{a(A\cap b)}} \cong$$

$$\cong \underset{H}{\boxed{A\cap B}}/\underset{H}{\boxed{(A\cap B)}}\cap\underset{K}{\boxed{a(A\cap b)}} = \underbrace{(A\cap B)/(a\cap B)b\cap a(A\cap b)}_{\text{symmetric.}}$$

as $(A\cap B)\cap a(A\cap b) = (a\cap B)b\cap a(A\cap b)$

$$\underset{B\cap a(A\cap b)}{\shortparallel} \qquad \supseteq \text{easy} \qquad \subseteq \downarrow$$

Given $\alpha\beta$, $\alpha\in a$, $\beta\in A\cap b$, $\alpha\beta\in B \Rightarrow \alpha\in B \Rightarrow \alpha\beta\in(a\cap B)\cap b$

Exercise. Use the same principle to show that any two finite bases of a vector space have the same cardinality.

Example. $\mathbb{Z}/n$ is simple iff n is prime.

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done line

Added oct 7, 2010. There's a much simpler proof of Jordan-Hölder in Etingof's "Groups around us":

(http://www-math.mit.edu/~etingof/groups.pdf)

Simple groups are important because any finite group can be decomposed into simple ones in a unique way, similarly to how a molecule can be decomposed into atoms. More precisely, we have the following theorem, called **the Jordan-Hölder theorem**.

Theorem 2.71. Let G be a finite group. Then there exists a sequence of subgroups $G = G_0 \supset G_1 \supset ... \supset G_n = \{e\}$ such that G_{i+1} is normal in G_i, and the groups $H_i := G_{i-1}/G_i$ are simple. Moreover, the sequence of groups $H_1, ..., H_n$, up to permutation, depends only on G and not on G_i.

Definition 2.72. The sequence $H_1, ..., H_n$ is called **the composition series** of G.

This theorem implies that if we understand finite simple groups, then to some extent we will understand all the finite groups. (With the understanding that this is not the full picture, since there are many complicated ways in which simple groups H_i can be "sown together" into G. This is similar to the distinction in chemistry between composition formula and structure formula of a substance).

Proof. To prove the existence of G_i, we can choose G_i to be a maximal normal subgroup in G_{i-1} (not equal G_{i-1} itself). Now we prove uniqueness of the composition series by induction in $|G|$. Assume that there are two collections of subgroups, G_i and G_i'. If $G_1 = G_1'$, the statement follows from the induction assumption. Otherwise, we have homomorphisms $f : G \to H_1$, $f' : G \to H_1'$, which combine into a surjective homomorphism $f'' : G \to H_1 \times H_1'$. Let K be the kernel of this homomorphism. Let $L_1, ..., L_r$ be the composition series of K (well defined by the induction assumption). Then G_1 has composition series

$$(H_2, ..., H_n) = (H_1', K_1, ..., K_r),$$

and G_1' has composition series

$$(H_2', ..., H_m') = (H_1, K_1, ..., K_r).$$

Thus, adding H_1 to the first series and H_1' to the second. we get

$$(H_1, H_2, ..., H_n) = (H_1', H_2', ..., H_m'),$$

as desired. □

Exercise 2.73. (i) Show that if H is a normal subgroup in G then the composition series of G is obtained by combining the composition series of H and G/H.

(ii) Show that if G is a group of order p^n, where p is a prime, then its composition series consists of n copies of $\mathbb{Z}_p$.

Exercise 2.74. Find the composition series of S_n.

Solution: For $n = 3$, $\mathbb{Z}_2, \mathbb{Z}_3$. For $n = 4$, three copies of $\mathbb{Z}_2$ and $\mathbb{Z}_3$. For $n \geq 5$, $\mathbb{Z}_2$ and A_n.

Definition 2.75. A finite group G is **solvable** if all its composition factors are cyclic.

Added April 4, 2011: Is this the same as the "Diamond Lemma" proof alluded to at http://sbseminar.wordpress.com/2009/11/20/the-diamond-lemma/?

The Symmetric Group.

Def $(-1)^\sigma = \text{sign}(\sigma) = \prod_{i<j} \text{sign}(\sigma(j) - \sigma(i))$

claim $(-1)^{\sigma\tau} = (-1)^\sigma (-1)^\tau$

pf $(-1)^{\sigma\tau} = \prod \text{sign}(\sigma\tau(j) - \sigma\tau(i)) =$

Pf $(-1)^{\sigma\tau} = \prod_{i<j} \text{sign}(\sigma\tau(j) - \sigma\tau(i)) =$

$$\prod_{i<j} \text{sign}(\tau(j) - \tau(i)) \prod_{i<j} \frac{\text{sign}(\sigma\tau(j) - \sigma\tau(i))}{\text{sign}(\tau(j) - \tau(i))} =$$

$$= \text{sign}(\tau) \cdot \text{sign}(\sigma)$$

"the alternating group"

So $\text{sign}: S_n \longrightarrow \{\pm 1\} = \mathbb{Z}/2$. Let $A_n = \ker \text{sign}$.

[A_n is the set of perms that can be written as an even product of transpositions]

Theorem. A_n is simple for $n \neq 4$.

Cycle Decomposition. $(12)(345) = [21453] = 21453$

Claim If $\sigma = (a_1 \ldots a_k)$ and $\tau = [\tau_1 \tau_2 \ldots \tau_n]$, then

$$\sigma^\tau = \tau^{-1}\sigma\tau = (\tau^{-1}(a_1), \tau^{-1}a_2, \ldots)$$

Corollary σ is conjugate to σ' iff they have the same cycle lengths

Corollary $\#(\text{Conjugacy classes of } S_n) = P(n)$

Lemma 1. Every element of A_n is a product of 3-cycles.

Pf $(12)(23) = (123)$, $(123)(234) = (12)(34)$ ....

Lemma 2. If $N \lhd A_n$ contains a 3-cycle, then $N = A_n$

Pf WLOG, $(123) \in N$. Claim For $\sigma \in S_n$, $(123)^\sigma \in N$ $\left(\begin{matrix}\sigma \in A_n \checkmark \\ \sigma = (12)\sigma' \checkmark\end{matrix}\right)$

So N contains all 3-cycles ... $\square$

Now take $N \lhd A_n$ w/ $N \neq \{1\}$

Case 1. N contains an element w/ cycle of length ≥ 4

$\sigma = (123456)\sigma' \in N$ $\sigma^{-1}(123)\sigma(123)^{-1} = (136)$

Case 2. N contains an element $\sigma = (123)(456)\sigma'$

consider $\sigma^{-1}(124)\sigma(124)^{-1}$

Case 3. N contains $\sigma = (123)$(product of pairs)

Then $\sigma^2 = (132)$ . . .

Case 4. Every element of N is a product of disjoint 2-cycles.

$\sigma = (12)(34)\sigma' \Rightarrow \sigma^{-1}(123)\sigma(123)^{-1} = (13)(24) = \tau \in N$

$\Rightarrow \tau^{-1}(125)\tau(125)^{-1} = (13452) \in N$