October 5, hours 10-11: A_n is Simple, G-Actions

September-30-10
12:19 PM

Read Along: Hungerford's book 7.10, Selick notes 1.7.

Reminder. 1. sign: $S_n \to \{\pm 1\}$, $A_n := \ker \text{sign}$.

2. Every permutation is a product of disjoint cycles.

3. Two such products are conjugate iff their length profiles are the same.

Theorem. A_n is simple for $n \neq 4$.

Lemma 1. Every element of A_n is a product of 3-cycles.

Pf $(12)(23) = (123)$, $(123)(234) = (12)(34)$ ...

Lemma 2. If $N \triangleleft A_n$ contains a 3-cycle, then $N = A_n$

Pf WLOG, $(123) \in N$. Claim For $\sigma \in S_n$, $(123)^\sigma \in N$ $\left(\begin{matrix}\sigma \in A_n \checkmark \\ \sigma = (12)\sigma' \checkmark\end{matrix}\right)$

So N contains all 3-cycles... $\square$

Now take $N \triangleleft A_n$ w/ $N \neq \{1\}$ [Always take the commutation w/ a 3-cycle]

Case 1. N contains an element w/ cycle of length ≥ 4

$\sigma = (123456)\sigma' \in N$ $\quad \sigma^{-1}(123)\sigma(123)^{-1} = (136)$

Case 2. N contains an element $\sigma = (123)(456)\sigma'$

Consider $\sigma^{-1}(124)\sigma(124)^{-1}$

Case 3. N contains $\sigma = (123)$ (product of pairs)

Then $\sigma^2 = (132)$ ...

Case 4. Every element of N is a product of disjoint 2-cycles.

$\sigma = (12)(34)\sigma' \Rightarrow \sigma^{-1}(123)\sigma(123)^{-1} = (13)(24) = \tau \in N$

$\Rightarrow \tau^{-1}(125)\tau(125)^{-1} = (13452) \in N$

Definition A G-set (left-G-set) $G \times X \to X$

s.t. $(g_1 g_2)x = g_1(g_2 x)$, $ex = x$. Same as $\alpha : G \to S(X)$.

G-sets are a category!

Examples. 1. G itself, under conjugation.

skipped {2. Subgroups(G), under conjugation.

Examples: 1. G/H when H is not-necessarily normal

skipped {Sub-example: S_n/S_{n-1} $\sigma S_{n-1} = \sigma' S_{n-1}$ iff

$\sigma(n) = \sigma'(n)$. Let $\tau_i(n) = i$, then

$\sigma \tau_i S_{n-1} = \tau_{\sigma i} S_{n-1}$. So S_n/S_{n-1} is $\{1 \dots n\}$ ...

2. If X_1, X_2 are G-sets, then so is $X_1 \sqcup X_2$.

Theorem. 1. Every G-set is a disjoint union of "transitive G-sets"

done live

2. If X is a transitive G set and $x \in X$, then $X \cong G/\mathrm{stab}_X(x)$. (So $|X| \mid |G|$)

Theorem. If X is a G set and x_i are representatives of the orbits, then

$$|X| = \sum_i \frac{|G|}{|\mathrm{Stab}_X(x_i)|}$$

Example. If G is a p-group, the centre of G is not empty.