October 28, hour 21: Left and right, Sylow apps, solvability

October-27-10
3:27 PM

On board:

$H < G$, $N \lhd G$, $H \cap N = \{e\}$
$\Rightarrow HN$ is a group

$h_1 n_1 h_2 n_2 = h_1 h_2 \phi_{h_2}(n_1) n_2$
with $\phi_h(n) = h^{-1} n h$

Read Along: Selick 2.1.

"A mirror flips left and right, yet not up and down; how can a mirror know???"

Dror is not a mirror!

Definition. Given abstract N, H & $\phi: H \to \mathrm{Aut}(N)$, the semi-direct product $H \ltimes N$...

Problem. $\phi: H \longrightarrow \mathrm{Aut}(N)$ is not a homomorphism!

$$\phi_a \circ \phi_b = \phi_{ba} \, !$$

Resolution. 1. Define "an anti-homomorphism".

2. Define H^{op} / $\mathrm{Aut}(N)^{op}$ [Exercise $g \mapsto g^{-1}$ is a homomorphism $G \to G^{op}$]

3. Talk about $N \rtimes H$, instead: $n_1 h_1 n_2 h_2 = n_1 n_2^{h_1^{-1}} h_1 h_2$

4. Talk about left/right G-actions, left-right cosets, and note that everything we've ever said about "conjugation action" was slightly wrong.

Groups of order 12. If $|G| = 12$, $P_4 = \mathbb{Z}/4$ or $(\mathbb{Z}/2)^2$, $P_3 = \mathbb{Z}/3$, and at least one of those is normal, for there's not enough room for 4 P_3 & 3 P_4's. So G is a semi-direct product.

$\mathbb{Z}/12$, $\mathbb{Z}/2 \times \mathbb{Z}/6$, A_4, D_{12}, $\mathbb{Z}/3 \rtimes \mathbb{Z}/4$

The most fun case is $\mathbb{Z}/3 \circlearrowright (\mathbb{Z}/2)^2$, giving A_4.

Solvable Groups. Def G is solvable if all quotients in its Jordan-Hölder series are Abelian.

Thm 1. If $N \lhd G$, G is solvable iff N & G/N are.

2. If $H < G$ and G is solvable, so is H.

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$A \triangleleft B$ $\quad H \cap A \triangleleft H \cap B$? ✓ $\frac{H \cap B}{H \cap A} \longrightarrow B/A$ by $[b]_{H \cap A} \to [b]_A$ is injective.

Done line

Rings.

Definition 2.1.1. *A* ***ring*** *consists of a set* R *together with binary operations* $+$ *and* $\cdot$ *satisfying:*

1. $(R, +)$ *forms an abelian group,*
2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\forall a, b, c \in R$,
3. $\exists 1 \neq 0 \in R$ *such that* $a \cdot 1 = 1 \cdot a = a$ $\forall a \in R$, *and*
4. $a \cdot (b + c) = a \cdot b + a \cdot c$ *and* $(a + b) \cdot c = a \cdot c + b \cdot c$ $\forall a, b, c \in R$.

Also define: Commutative ring.

Examples. $\mathbb{Z}$, $R[x]$, $M_{n \times n}(R)$

Morphism, im, subring, ker, ideal.

Q. Is every ideal a quotient.

Ans. Define R/I.