Greg's way

November-13-10
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26 I reject the premise of the question. :-)

It is true, as Terry suggests, that there is a nice dynamical proof of the classification of finite abelian groups. If A is finite, then for every prime p has a stable kernel A_p and a stable image A^p in A, by definition the limits of the kernel and image of p^n as $n \to \infty$. You can show that this yields a direct sum decomposition of A, and you can use linear algebra to classify the dynamics of the action of p on A_p. A similar argument appears in Matthew Emerton's proof. As Terry says, this proof is nice because it works for finitely generated torsion modules over any PID. In particular, it establishes Jordan canonical form for finite-dimensional modules over $k[x]$, where k is an algebraically closed field. My objection is that finite abelian groups look easier than finitely generated abelian groups in this question.

The slickest proof of the classification that I know is one that assimilates the ideas of Smith normal form. Ben's question is not entirely fair to Smith normal form, because you do not need finitely many relations. That is, Smith normal form exists for matrices with finitely many columns, not just for finite matrices. This is one of the tricks in the proof that I give next.

Theorem. If A is an abelian group with n generators, then it is a direct sum of at most n cyclic groups.

Proof. By induction on n. If A has a presentation with n generators and no relations, then A is free and we are done. Otherwise, define the height of any n-generator presentation of A to be the least norm $|x|$ of any non-zero coefficient x that appears in some relation. Choose a presentation with least height, and let $a \in A$ be the generator such that $R = xa + \ldots = 0$ is the pivotal relation. (Pun intended. :-))

The coefficient y of a in any other relation must be a multiple of x, because otherwise if we set $y = qx + r$, we can make a relation with coefficient r. By the same argument, we can assume that a does not appear in any other relation.

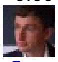
The coefficient z of another generator b in the relation R must also be a multiple of x, because otherwise if we set $z = qx + r$ and replace a with $a' = a + qb$, the coefficient r would appear in R. By the same argument, we can assume that the relation R consists only of the equation $xa = 0$, and without ruining the previous property that a does not appear in other relations. Thus $A \cong \mathbb{Z}/x \oplus A'$, and A' has $n-1$ generators. □

Compare the complexity of this argument to the other arguments supplied so far.

Minimizing the norm $|x|$ is a powerful step. With just a little more work, you can show that x divides every coefficient in the presentation, and not just every coefficient in the same row and column. Thus, each modulus x_k that you produce divides the next modulus x_{k+1}.

Another way to describe the argument is that Smith normal form is a matrix version of the Euclidean algorithm. If you're happy with the usual Euclidean algorithm, then you should be happy with its matrix form; it's only a bit more complicated. The proof immediately works for any Euclidean domain; in particular, it also implies the Jordan canonical form theorem. And it only needs minor changes to apply to general PIDs.

link|flag|cite edited Jan 17 at 3:25 answered Jan 17 at 0:09



Greg Kuperberg
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I happened to be looking at M.A. Armstrong's very nice book "Groups and Symmetry" today and noticed that he has a proof along similar lines, except that he first minimizes the number of generators and then the height. Minimizing the number of generators seems to be necessary in order to obtain a splitting where each modulus divides the next. (Perhaps minimizing the number of generators is implicit in the "By induction on n" at the beginning of Greg's proof.) – Allen Hatcher Jan 25 at 22:10

I think that it's okay as it is. If you have more than the minimum number of generators, then I think that the arguments yields summands that are just Z/1. – Greg Kuperberg Jan 26 at 4:01

You're right, the “just a little more work” takes care of it. – Allen Hatcher Jan 26 at 8:34

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