Boxing Day Scratch

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9:12 AM

$$\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \mapsto \begin{pmatrix} 2-x & 0 \\ 1 & 2-x \end{pmatrix} \to \begin{pmatrix} 1 & 2-x \\ 2-x & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2-x \\ 0 & -(2-x)^2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & (2-x)^2 \end{pmatrix}$$

If $\gcd(a,b)=1$ $\quad 1=sa+tb$ $\quad$ Then

$$\frac{R}{\langle a\rangle} \oplus \frac{R}{\langle b\rangle} \cong \frac{R}{\langle ab\rangle}$$ via

$$\begin{tikzcd} R/\langle a\rangle \arrow[dr, "t\cdot b"] & & & R/\langle a\rangle \\ \oplus & R/\langle ab\rangle \arrow[urr, "1"] \arrow[drr, "1"'] & & \oplus \\ R/\langle b\rangle \arrow[ur, "s\cdot a"'] & & & R/\langle b\rangle \end{tikzcd}$$

$$\begin{tikzcd} R^2 \arrow[r, "\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}"] \arrow[d, "\begin{pmatrix} a & -b \\ t & s \end{pmatrix}"'] & R^2 \arrow[d, "\begin{pmatrix} 1 & -1 \\ tb & sa \end{pmatrix}"] \\ R^2 \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix}"'] & R^2 \end{tikzcd}$$

$$\begin{pmatrix} 1 & -1 \\ tb & sa \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & -b \\ tab & sab \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix}\begin{pmatrix} a & -b \\ t & s \end{pmatrix} = \begin{pmatrix} a & -b \\ tab & sab \end{pmatrix} \quad ✓$$

$$\begin{pmatrix} s & -t \\ b & a \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} as & -tb \\ ab & ab \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix}\begin{pmatrix} as & -tb \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} as & -tb \\ ab & ab \end{pmatrix}$$

Theorem. If $(b,c)=1$ & $(a,c)=1$ Then $(ab,c)=1$

Proof. $sb+tc=1$, $ua+vc=1 \Rightarrow sb(ua+vc)+tc=1$

so $\quad us\cdot ab+(sbv+t)c=1$ $\quad$ } conceptual meaning?

$x-\lambda, x-\mu$ $\quad \frac{1}{\mu-\lambda}(x-\lambda)-\frac{1}{\mu-\lambda}(x-\mu)=1$ $\quad$ so $\begin{pmatrix} \frac{1}{\lambda-\mu} & \frac{1}{\lambda-\mu} \\ x-\lambda & x-\mu \end{pmatrix}$ is invertible.

Question Assume for invertible $P, Q \in M_{n\times n}(R)$,

$$P(Ix-A)Q = Ix-B \qquad (B \text{ diagonal, distinct entries})$$

Does it follow that $P=Q\in M_{n\times n}(F)$ and $PQ=I$?

Probably not, but it should be possible to replace P,Q with P', Q' that do have this property.

Problem Consider $(x-\lambda_i)$, $i=1,\dots,n$, λ_i distinct. Find an invertible . . . . .

$$\begin{pmatrix} x-\lambda_1 & 0 \\ 0 & x-\lambda_2 \end{pmatrix} \mapsto \begin{pmatrix} x-\lambda_1 & x-\lambda_2 \\ 0 & x-\lambda_2 \end{pmatrix} \mapsto \begin{pmatrix} \lambda_2-\lambda_1 & x-\lambda_2 \\ \lambda_2-x & x-\lambda_2 \end{pmatrix}$$

$\mapsto$

Claim $\dfrac{R}{\langle (x-\lambda_1)\cdots(x-\lambda_n)\rangle} \cong \bigoplus \dfrac{R}{\langle x-\lambda_i\rangle} \cong \bigoplus F$

Δ !!

$$[f] \mapsto ([f],[f],\dots [f]) \to (f(\lambda_1),\dots,f(\lambda_n))$$

$$\Big[\sum c_i l_i(x)\Big] := \Big[\sum c_i \frac{\prod_{j\neq i}(x-\lambda_j)}{\prod_{j\neq i}(\lambda_i-\lambda_j)}\Big] \longleftarrow\!\!| \ (c_1,\dots,c_n)$$

$$\begin{tikzcd}
R^1 \arrow[r, "\cdot\prod(x-\lambda_i)=\Delta"] \arrow[d, shift right, "0"'] & R^1 \arrow[r] \arrow[d, shift right, "\phi"'] & M \arrow[r] \arrow[d, shift right, "\phi"'] & 0 \\
R^n \arrow[r, "Ix-B"', "=\begin{pmatrix} x-\lambda_1 & & 0 \\ & \ddots & \\ 0 & & x-\lambda_n\end{pmatrix}"' below] \arrow[u, shift right, "\psi_1"'] & R^n \arrow[r] \arrow[u, shift right, "\psi"'] \arrow[l, bend right, "h"'] & M \arrow[r] \arrow[u, shift right, "\bar\psi"'] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
R^r \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & R^g \arrow[r] \arrow[d, shift right] \arrow[d, shift left] \arrow[dl] & M \arrow[r] & 0 \\
R^{r'} \arrow[r] & R^{g'} \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

$\phi: f \mapsto (f(\lambda_1),\dots,f(\lambda_n))$

$\psi: (g_1\dots g_n) \longmapsto \sum g_i l_i \qquad \psi_1(g_1\dots g_n) = \sum_{i=1}^n \dfrac{g_i}{\prod_{j\neq i}(\lambda_i-\lambda_j)}$

$$h(g_1,\dots g_n) = \left(\frac{g_1(x)-g_1(\lambda_1)}{x-\lambda_1}, \dots, \frac{g_n(x)-g_n(\lambda_n)}{x-\lambda_n}\right)$$

Question When and how can a "homotopy isomorphism" be upgraded to a "change of basis isomorphism"?

$$\begin{tikzcd}
R^n \arrow[r, "\begin{pmatrix}1 & & \\ & 1 & \\ & & \Delta\end{pmatrix}"] & R^n \arrow[r] \arrow[d, shift right, "\phi"'] & M \arrow[r] & 0 \\
R^n \arrow[r, "Ix-B"', "=\begin{pmatrix}x-\lambda_1 & & \\ & \ddots & \\ & & x-\lambda_n\end{pmatrix}"' below] & R^n \arrow[r] \arrow[u, shift right, "\psi"'] & M \arrow[r] & 0
\end{tikzcd}$$

$\phi: (f_1,\dots,f_n) \longmapsto (f_n + (x-\lambda_1)f_1, \dots, f_n + (x-\lambda_{n-1})f_{n-1}, f_n)$ X

Aside Which linear combination of $\Delta_{n-1} = \prod_1^{n-1}(x-\lambda_i)$ and $(x-\lambda_n)$ is 1?

Sol'n $\dfrac{1}{\Delta_{n-1}(\lambda_n)}\left[\Delta_{n-1}(x) - \dfrac{\Delta_{n-1}(x)-\Delta_{n-1}(\lambda_n)}{x-\lambda_n}(x-\lambda_n)\right] = 1$

$\Delta_{n-1}(\lambda_n) \Big[\ldots_{n-1}(\ldots) \qquad x-\lambda_n \qquad (\ldots-\lambda_n) \Big]$

<u>Aside</u>

$$A=\begin{pmatrix} p & 0 & 0 \\ 1 & p & 0 \\ 0 & 1 & p \end{pmatrix} \mapsto \begin{pmatrix} 1 & p & 0 \\ 0 & 1 & p \\ p & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & p & 0 \\ 0 & 1 & p \\ 0 & -p^2 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & p \\ 0 & p^2 & 0 \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & p \\ 0 & 0 & -p^3 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^3 \end{pmatrix} = B$$

(Better to see abstractly that the module with presentation matrix A is isomorphic to the module with presentation matrix B).

<u>Claim</u> The quotients

$$\langle x_1, \ldots, x_s \mid \underset{i<s}{px_i = x_{i+1}},\ px_s = 0 \rangle \ \&\ \langle y_1 \ldots y_s \mid \begin{matrix} y_i = 0 & i<s \\ py_s = 0 & \end{matrix} \rangle$$

are "ambient isomorphic".

<u>proof</u>

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_s \end{pmatrix} \mapsto \begin{pmatrix} y_s \\ py_s - y_1 \\ p^2 y_s - y_2 - py_1 \\ \vdots \\ p^{s-1} y_s \ \vdots \end{pmatrix} \qquad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_s \end{pmatrix} \mapsto \begin{pmatrix} px_1 - x_2 \\ px_2 - x_3 \\ px_3 - x_2 \\ \vdots \\ px_{s-1} - x_s \\ x_1 \end{pmatrix}$$