Pensieve header: A messy algebraic number.

From https://drorbn.net/index.php?title=08-401/Homework_Assignment_8:

Just for Fun [edit]

Let x be some root of the equation

$$x^5+(\sqrt[3]{2}-\sqrt{3})x^4+\frac{1}{\sqrt{\sqrt{3}+\sqrt{5}}}x^3-1=0.$$

We know that x is algebraic. Can you find a polynomial f with rational coefficients whose roots include x?

$$\mathtt{Solve}\left[x^5+\left(\sqrt[3]{2}-\sqrt[2]{3}\right)x^4+\frac{\sqrt{7}}{\sqrt{\sqrt{3}+\sqrt{5}}}\,x^3-1==0,\ x\right][[1,1,2,1]][x]$$

$$64 - 1536 x^5 - 2304 x^8 + 1536 x^{10} - 38656 x^{12} + 50688 x^{13} + 209152 x^{15} + 38016 x^{16} + 774144 x^{17} - 48384 x^{18} - 430464 x^{20} - 760320 x^{21} - 1502592 x^{22} - 5644800 x^{23} + 6214960 x^{24} - 22262784 x^{25} + 691200 x^{26} - 55336064 x^{27} + 23463936 x^{28} - 97790208 x^{29} + 120194560 x^{30} + 67703040 x^{31} + 303607872 x^{32} + 270618624 x^{33} + 165514752 x^{34} + 551579904 x^{35} - 910449920 x^{36} + 895728384 x^{37} - 1915432704 x^{38} + 3934938240 x^{39} - 5845936896 x^{40} + 5594910720 x^{41} - 13991680128 x^{42} - 592952832 x^{43} - 40607931840 x^{44} + 13071701248 x^{45} - 34550117568 x^{46} + 60965547264 x^{47} + 47107596684 x^{48} + 210223102464 x^{49} + 50603096064 x^{50} + 187250131392 x^{51} - 127218790656 x^{52} - 220165158240 x^{53} - 875237095680 x^{54} - 439653846144 x^{55} - 1029596693232 x^{56} + 77560008704 x^{57} + 426104198112 x^{58} + 2914022225280 x^{59} + 1244754080592 x^{60} + 3957273835872 x^{61} - 77816699136 x^{62} + 211491443216 x^{63} - 7082856051480 x^{64} - 1240435229952 x^{65} - 9099760732240 x^{66} + 1479763885440 x^{67} - 40166818392 x^{68} + 11802294150880 x^{69} + 430739318184 x^{70} + 11347109944416 x^{71} - 4874437765615 x^{72} - 4656848052288 x^{73} - 14909227451328 x^{74} - 2673515984840 x^{75} - 10864630423872 x^{76} + 7310305053984 x^{77} + 9402782657760 x^{78} + 15685489060560 x^{79} + 7753119259740 x^{80} + 8163353004800 x^{81} - 4160438045136 x^{82} - 7792772074032 x^{83} - 10837871323064 x^{84} - 6351752611776 x^{85} - 2653267912656 x^{86} + 3406043294720 x^{87} + 3854093230140 x^{88} + 2979299273472 x^{89} + 682028554624 x^{90} - 467338215744 x^{91} - 1318617095616 x^{92} - 419605864704 x^{93} - 175270257024 x^{94} + 215291179392 x^{95} + 139321525600 x^{96} + 61079001600 x^{97} - 36749603712 x^{98} - 23550230912 x^{99} - 8608419264 x^{100} + 5362157568 x^{101} + 7142681856 x^{102} + 1546369536 x^{103} - 2217779472 x^{104} - 1107994624 x^{105} + 354114048 x^{106} + 313642368 x^{107} - 12473600 x^{108} - 56452608 x^{109} - 8620032 x^{110} + 6463744 x^{111} + 2414784 x^{112} - 350208 x^{113} - 372992 x^{114} - 3840 x^{115} + 38016 x^{116} + 1024 x^{117} - 2304 x^{118} + 64 x^{120}$$

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PolOf[x_?NumberQ] := # - x;
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RootsOf /: RootsOf[p_]^Rational[1, n_] := RootsOf[p /. # → #^n];
RootsOf /: Sqrt[RootsOf[p_]] := RootsOf[p /. # → #^2];
a_ * RootsOf[p_] /; FreeQ[a, #] ^:= RootsOf[p /. {# → 1 / a * #}];
a_ + RootsOf[p_] /; FreeQ[a, #] ^:= RootsOf[Expand[p /. # → # - a]];
RootsOf /: RootsOf[p_]^(-1) := RootsOf[Expand[1 / #^Exponent[p, #] p] /. # → 1 / #];
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RootsOf /: RootsOf[a_] * RootsOf[b_] := Module[
   {x, y, pow, dx, dy, rulex, ruley, mat},
   pow = 1 / (x y);
   px = a /. # → x; py = b /. # → y;
   dx = Exponent[px, x];
   dy = Exponent[py, y];
   rulex = x^dx → Expand[-1 / Coefficient[px, x^dx] * px + x^dx];
   ruley = y^dy → Expand[-1 / Coefficient[py, y^dy] * py + y^dy];
   mat = Table[
     pow *= x * y;
     pow = Expand[Expand[pow] /. {rulex, ruley}];
     Flatten[PadRight[CoefficientList[pow, {x, y}], {dx, dy}]],
     {k, 0, dx * dy}
    ];
   t = NullSpace[Transpose[mat]];
   RootsOf[Sum[t⟦1, k + 1⟧ #^k, {k, 0, dx dy}]]
  ];
RootsOf /: RootsOf[a_] + RootsOf[b_] := Module[
   {x, y, pow, dx, dy, rulex, ruley, mat},
   pow = 1 / (x + y);
   px = a /. # → x; py = b /. # → y;
   dx = Exponent[px, x];
   dy = Exponent[py, y];
   rulex = x^dx → Expand[-1 / Coefficient[px, x^dx] * px + x^dx];
   ruley = y^dy → Expand[-1 / Coefficient[py, y^dy] * py + y^dy];
   mat = Table[
     pow *= x + y;
     pow = Expand[Expand[pow] /. {rulex, ruley}];
     Flatten[PadRight[CoefficientList[pow, {x, y}], {dx, dy}]],
     {k, 0, dx * dy}
    ];
   t = NullSpace[Transpose[mat]];
   RootsOf[Sum[t⟦1, k + 1⟧ #^k, {k, 0, dx dy}]]
  ];
RootsOf /: RootsOf[p_]^n_Integer /; n > 1 := RootsOf[p] * RootsOf[p]^(n - 1);

RootsOf[p_ /; ! FreeQ[p, RootsOf]] := Module[
   {cl, RulePol, RulesCoef = {}, mat, a, pow = 1 / #, poldeg = Exponent[p, #], t},
   vars = {#};
   vardegs = {poldeg};
   cl = CoefficientList[p, #];
   cl = cl / Last[cl];
   Do[
    If[Head[cl⟦i⟧] === RootsOf,
     AppendTo[RulesCoef,
      pa = First[cl⟦i⟧] /. # → a[i - 1];
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        pad = Exponent[pa, a[i - 1]];
        a[i - 1] ^pad → Expand[-1 / Coefficient[pa, a[i - 1], pad] * pa + a[i - 1] ^pad]
       ];
       AppendTo[vars, a[i - 1]];
       AppendTo[vardegs, pad],
       a[i - 1] = cl[[i]]
      ], {i, poldeg}
     ];
    RulePol = #^poldeg → -Sum[#^i a[i], {i, 0, poldeg - 1}];
    mat = Table[
      pow *= #;
      pow = Expand[Expand[pow] /. RulePol /. RulesCoef];
      Flatten[PadRight[CoefficientList[pow, vars], vardegs]],
      {k, 0, Times @@ vardegs}
     ];
    t = NullSpace[Transpose[mat]];
    RootsOf[Sum[t[[1, k + 1]] #^k, {k, 0, Times @@ vardegs}]]
   ];
{two, three, five, seven} = RootsOf[PolOf[#]] & /@ {2, 3, 5, 7}
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{RootsOf[-2 + #1], RootsOf[-3 + #1], RootsOf[-5 + #1], RootsOf[-7 + #1]}

$\sqrt[3]{\text{two}}$

RootsOf$\left[-2+\#1^3\right]$

$\sqrt[3]{\text{two}} - \sqrt[2]{\text{three}}$

RootsOf$\left[-23-36\,\#1+27\,\#1^2-4\,\#1^3-9\,\#1^4+\#1^6\right]$

$$\frac{1}{\sqrt{\sqrt{\text{three}}+\sqrt{\text{five}}}}$$

RootsOf$\left[1-16\,\#1^4+4\,\#1^8\right]$

$$f = \text{First}\left[\text{RootsOf}\left[\#^5+\left(\sqrt[3]{\text{two}}-\sqrt[2]{\text{three}}\right)\#^4+\frac{1}{\sqrt{\sqrt{\text{three}}+\sqrt{\text{five}}}}\#^3-1\right]\right] \;/.\; \# \to x$$

$$1-48x^{5}-72x^{8}+1128x^{10}-56x^{12}+3312x^{13}-17296x^{15}+2484x^{16}+2496x^{17}-74520x^{18}+197316x^{20}-109296x^{21}-54384x^{22}+1092960x^{23}-\frac{108237x^{24}}{2}-1828944x^{25}+2349864x^{26}+771936x^{27}-11806992x^{28}+2273268x^{29}+14699784x^{30}-32898096x^{31}-7218384x^{32}+101043504x^{33}-46607718x^{34}-106526160x^{35}+337832742x^{36}+32821068x^{37}-721088568x^{38}+621509340x^{39}+695017359x^{40}-2722078464x^{41}+201338370x^{42}+4456598400x^{43}-6062094183x^{44}-3876725684x^{45}+17998401708x^{46}-5635663080x^{47}-\frac{390661395025x^{48}}{16}+46129721700x^{49}+16389802878x^{50}-100900796888x^{51}+62948764056x^{52}+118953927030x^{53}-285327820842x^{54}-33424403460x^{55}+\frac{982680094089x^{56}}{2}-487208787980x^{57}-503805549633x^{58}+1476251331444x^{59}-193282449778x^{60}-2110733432226x^{61}+2927514504030x^{62}+$$

$$1\,749\,977\,416\,998\,x^{63} - 6\,526\,412\,313\,489\,x^{64} + 2\,677\,036\,864\,872\,x^{65} + 8\,033\,796\,861\,441\,x^{66} -$$
$$14\,323\,577\,057\,328\,x^{67} - \frac{8\,560\,004\,815\,515\,x^{68}}{2} + 25\,054\,660\,833\,522\,x^{69} - 18\,086\,469\,856\,590\,x^{70} -$$
$$26\,803\,294\,491\,348\,x^{71} + \frac{937\,274\,831\,377\,493\,x^{72}}{16} + 2\,276\,586\,880\,518\,x^{73} - 84\,461\,414\,843\,379\,x^{74} +$$
$$88\,859\,798\,962\,908\,x^{75} + 75\,930\,688\,867\,662\,x^{76} - \frac{407\,146\,578\,366\,243\,x^{77}}{2} + 45\,261\,379\,200\,729\,x^{78} +$$
$$251\,154\,192\,400\,986\,x^{79} - 346\,234\,827\,959\,049\,x^{80} - 170\,058\,140\,000\,874\,x^{81} + \frac{2\,430\,199\,794\,539\,325\,x^{82}}{4} -$$
$$302\,714\,566\,768\,554\,x^{83} - \frac{2\,620\,562\,454\,373\,805\,x^{84}}{4} + \frac{2\,207\,598\,782\,655\,363\,x^{85}}{2} + 243\,554\,298\,340\,497\,x^{86} -$$
$$\frac{3\,130\,773\,381\,073\,217\,x^{87}}{2} + \frac{9\,810\,028\,731\,026\,607\,x^{88}}{8} + 1\,470\,425\,684\,953\,476\,x^{89} -$$
$$\frac{11\,626\,910\,892\,009\,743\,x^{90}}{4} + 59\,873\,972\,728\,032\,x^{91} + \frac{27\,787\,926\,050\,034\,465\,x^{92}}{8} -$$
$$\frac{7\,322\,929\,121\,130\,961\,x^{93}}{2} - \frac{5\,479\,891\,043\,201\,097\,x^{94}}{2} + 6\,272\,531\,663\,800\,803\,x^{95} -$$
$$\frac{409\,405\,210\,323\,616\,177\,x^{96}}{256} - \frac{13\,073\,432\,181\,182\,547\,x^{97}}{2} + \frac{33\,030\,237\,155\,633\,451\,x^{98}}{4} +$$
$$4\,047\,750\,710\,716\,593\,x^{99} - 10\,664\,082\,705\,405\,840\,x^{100} + \frac{80\,712\,945\,791\,683\,065\,x^{101}}{16} +$$
$$\frac{40\,337\,313\,679\,133\,959\,x^{102}}{4} - \frac{26\,040\,500\,880\,109\,317\,x^{103}}{2} - \frac{157\,563\,272\,883\,906\,207\,x^{104}}{32} +$$
$$\frac{24\,751\,862\,794\,819\,753\,x^{105}}{2} - \frac{226\,512\,010\,963\,951\,047\,x^{106}}{32} - \frac{24\,453\,005\,851\,264\,743\,x^{107}}{2} +$$
$$\frac{258\,259\,201\,379\,114\,353\,x^{108}}{32} + \frac{143\,543\,719\,220\,239\,611\,x^{109}}{16} - \frac{9\,181\,516\,661\,881\,245\,x^{110}}{4} -$$
$$\frac{142\,401\,893\,148\,243\,693\,x^{111}}{16} + \frac{851\,381\,420\,274\,774\,693\,x^{112}}{64} + \frac{123\,041\,854\,824\,209\,031\,x^{113}}{4} -$$
$$\frac{1\,159\,110\,319\,447\,597\,455\,x^{114}}{32} - 31\,854\,901\,553\,636\,202\,x^{115} + \frac{5\,488\,718\,276\,058\,939\,045\,x^{116}}{64} -$$
$$\frac{479\,474\,714\,731\,248\,531\,x^{117}}{16} - \frac{2\,270\,748\,530\,889\,698\,187\,x^{118}}{16} + \frac{1\,147\,407\,127\,184\,872\,359\,x^{119}}{8} +$$
$$\frac{45\,504\,455\,216\,963\,114\,995\,x^{120}}{512} - \frac{4\,755\,126\,316\,621\,188\,177\,x^{121}}{16} + \frac{3\,939\,289\,469\,968\,676\,337\,x^{122}}{32} +$$
$$\frac{2\,853\,382\,225\,146\,440\,015\,x^{123}}{8} - \frac{6\,978\,650\,849\,143\,146\,873\,x^{124}}{16} - \frac{7\,363\,344\,128\,835\,678\,951\,x^{125}}{64} +$$
$$\frac{23\,053\,685\,051\,169\,378\,621\,x^{126}}{32} - \frac{6\,746\,718\,668\,706\,548\,055\,x^{127}}{16} - \frac{10\,285\,139\,825\,969\,096\,223\,x^{128}}{16} +$$
$$\frac{16\,613\,172\,530\,141\,421\,879\,x^{129}}{16} - \frac{5\,025\,183\,104\,240\,729\,655\,x^{130}}{128} - \frac{21\,719\,004\,419\,602\,487\,385\,x^{131}}{16} +$$

$$\frac{140\,083\,738\,007\,628\,886\,359\,x^{132}}{128}+\frac{53\,284\,534\,595\,534\,354\,967\,x^{133}}{64}-\frac{63\,696\,650\,947\,382\,701\,179\,x^{134}}{32}+$$

$$\frac{38\,569\,298\,099\,284\,144\,475\,x^{135}}{64}+\frac{519\,134\,316\,856\,583\,877\,603\,x^{136}}{256}-\frac{8\,902\,695\,503\,036\,912\,511\,x^{137}}{4}-$$

$$\frac{79\,110\,769\,031\,497\,868\,307\,x^{138}}{128}+\frac{6\,216\,760\,747\,864\,877\,337\,x^{139}}{2}-\frac{444\,124\,323\,960\,121\,945\,899\,x^{140}}{256}-$$

$$\frac{149\,697\,887\,160\,270\,943\,473\,x^{141}}{64}+\frac{232\,658\,149\,031\,025\,103\,335\,x^{142}}{64}-\frac{8\,973\,221\,308\,514\,753\,049\,x^{143}}{32}-$$

$$\frac{16\,100\,311\,575\,067\,271\,732\,191\,x^{144}}{4096}+\frac{210\,617\,277\,643\,803\,135\,117\,x^{145}}{64}+\frac{239\,530\,635\,231\,234\,189\,291\,x^{146}}{128}-$$

$$\frac{153\,888\,677\,051\,396\,886\,599\,x^{147}}{32}+\frac{57\,855\,144\,449\,263\,309\,713\,x^{148}}{32}+\frac{250\,857\,433\,374\,501\,188\,601\,x^{149}}{64}-$$

$$\frac{603\,827\,030\,001\,620\,530\,873\,x^{150}}{128}-\frac{35\,391\,264\,886\,351\,280\,565\,x^{151}}{64}+\frac{2\,623\,339\,114\,814\,949\,873\,903\,x^{152}}{512}-$$

$$\frac{218\,279\,622\,019\,055\,739\,951\,x^{153}}{64}-\frac{369\,104\,002\,857\,803\,031\,981\,x^{154}}{128}+\frac{339\,764\,772\,702\,404\,708\,241\,x^{155}}{64}-$$

$$\frac{283\,664\,239\,957\,911\,741\,051\,x^{156}}{256}-\frac{137\,940\,514\,540\,087\,470\,375\,x^{157}}{32}+\frac{551\,433\,350\,603\,990\,679\,723\,x^{158}}{128}+$$

$$\frac{20\,108\,863\,392\,088\,158\,489\,x^{159}}{16}-\frac{2\,420\,306\,608\,180\,012\,918\,143\,x^{160}}{512}+\frac{9\,181\,932\,611\,974\,332\,609\,x^{161}}{4}+$$

$$\frac{22\,001\,459\,108\,374\,938\,243\,x^{162}}{8}-\frac{65\,246\,591\,608\,113\,627\,099\,x^{163}}{16}+\frac{10\,077\,621\,607\,581\,189\,369\,x^{164}}{64}+$$

$$\frac{52\,940\,260\,716\,474\,762\,237\,x^{165}}{16}-\frac{20\,101\,731\,768\,776\,678\,763\,x^{166}}{8}-\frac{19\,455\,273\,573\,240\,277\,587\,x^{167}}{16}+$$

$$\frac{382\,328\,643\,545\,477\,879\,577\,x^{168}}{128}-\frac{13\,047\,679\,600\,714\,863\,663\,x^{169}}{16}-\frac{57\,393\,396\,023\,995\,259\,799\,x^{170}}{32}+$$

$$\frac{30\,914\,362\,724\,535\,834\,141\,x^{171}}{16}+\frac{17\,183\,962\,849\,459\,632\,513\,x^{172}}{64}-\frac{13\,681\,042\,351\,759\,603\,263\,x^{173}}{8}+$$

$$\frac{24\,921\,194\,096\,486\,759\,403\,x^{174}}{32}+739\,430\,927\,672\,340\,840\,x^{175}-\frac{285\,536\,013\,085\,486\,718\,865\,x^{176}}{256}+$$

$$\frac{132\,931\,929\,504\,956\,331\,x^{177}}{2}+\frac{3\,075\,770\,135\,189\,718\,681\,x^{178}}{4}-462\,678\,576\,982\,582\,524\,x^{179}-$$

$$\frac{922\,639\,794\,964\,975\,247\,x^{180}}{4}+\frac{990\,937\,343\,772\,081\,537\,x^{181}}{2}-\frac{333\,021\,282\,218\,195\,631\,x^{182}}{4}-$$

$$\frac{547\,794\,921\,535\,111\,585\,x^{183}}{2}+\frac{3\,113\,430\,434\,224\,877\,931\,x^{184}}{16}+\frac{116\,445\,842\,085\,259\,137\,x^{185}}{2}-$$

$$\frac{684\,528\,685\,895\,239\,393\,x^{186}}{4}+\frac{68\,236\,355\,387\,442\,105\,x^{187}}{2}+\frac{631\,037\,957\,146\,741\,653\,x^{188}}{8}-$$

$$59\,651\,452\,210\,973\,535\,x^{189}-\frac{60\,249\,154\,624\,212\,789\,x^{190}}{4}+45\,862\,977\,319\,497\,678\,x^{191}-$$

$$\frac{110\,532\,591\,900\,814\,233\,x^{192}}{16} - 18\,913\,553\,418\,398\,220\,x^{193} + 12\,939\,569\,909\,975\,292\,x^{194} +$$
$$4\,619\,007\,017\,584\,014\,x^{195} - \frac{18\,530\,714\,802\,816\,669\,x^{196}}{2} + 94\,505\,021\,680\,110\,x^{197} +$$
$$3\,749\,776\,378\,411\,750\,x^{198} - 1\,702\,560\,810\,251\,466\,x^{199} - \frac{4\,858\,962\,701\,877\,987\,x^{200}}{4} +$$
$$1\,300\,402\,442\,381\,290\,x^{201} + 317\,492\,532\,465\,507\,x^{202} - 556\,253\,690\,450\,478\,x^{203} + \frac{94\,491\,734\,370\,319\,x^{204}}{2} +$$
$$201\,282\,529\,993\,632\,x^{205} - 98\,947\,938\,712\,509\,x^{206} - 75\,722\,334\,330\,392\,x^{207} + \frac{851\,375\,427\,208\,911\,x^{208}}{16} +$$
$$23\,667\,860\,391\,060\,x^{209} - 19\,591\,111\,192\,394\,x^{210} - 3\,655\,466\,884\,584\,x^{211} + 7\,016\,487\,757\,260\,x^{212} -$$
$$874\,935\,060\,844\,x^{213} - 2\,850\,979\,891\,746\,x^{214} + 757\,923\,897\,756\,x^{215} + \frac{2\,284\,882\,676\,955\,x^{216}}{2} -$$
$$254\,223\,845\,580\,x^{217} - 390\,948\,323\,970\,x^{218} + 51\,702\,050\,804\,x^{219} + 109\,346\,249\,841\,x^{220} -$$
$$5\,720\,547\,480\,x^{221} - 24\,988\,168\,514\,x^{222} - 180\,393\,456\,x^{223} + \frac{9\,404\,867\,073\,x^{224}}{2} + 222\,776\,576\,x^{225} -$$
$$732\,600\,576\,x^{226} - 51\,553\,968\,x^{227} + 94\,449\,092\,x^{228} + 7\,341\,456\,x^{229} - 9\,973\,824\,x^{230} - 709\,616\,x^{231} +$$
$$841\,098\,x^{232} + 45\,648\,x^{233} - 54\,168\,x^{234} - 1776\,x^{235} + 2484\,x^{236} + 32\,x^{237} - 72\,x^{238} + x^{240}$$

x0 =

$$\mathbf{x\ /.\ First\Big[NSolve\Big[x^5 + \left(\sqrt[3]{2} - \sqrt[2]{3}\right) x^4 + \frac{1}{\sqrt{\sqrt{3}+\sqrt{5}}}\,x^3 - 1 == 0,\ x,\ WorkingPrecision \to 100\Big]\Big]}$$

$-0.65598920670311749911258970723212291126332655070704125749960248032568807043455164361810\
50743691270111 -$
$0.60946130264350483354450547686838697655046371018265450410123538950945181929534121874894\
97407065330107\, i$

f /. x → x0

$0. \times 10^{-84} + 0. \times 10^{-84}\, i$