Pensieve header: An analysis of ${\mathfrak g}^\epsilon$ within gl_2.

1-Smidgen sl_2 Let $\mathfrak{g}_1$ be the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b,c,u,w\rangle$ over the ring $R=\mathbb{Q}[\epsilon]/(\epsilon^2=0)$, with b central and with $[w,c]=w$, $[c,u]=u$, and $[u,w]=b-2\epsilon c$, with CYBE $r_{ij}=(b_i-\epsilon c_i)c_j+u_iw_j$ in $\mathcal{U}(\mathfrak{g}_1)^{\otimes\{i,j\}}$. Over $\mathbb{Q}$, $\mathfrak{g}_1$ is a solvable approximation of sl_2: $\mathfrak{g}_1 \supset \langle b,u,w,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle \supset \langle b,\epsilon b,\epsilon c,\epsilon u,\epsilon w\rangle \supset 0$. (note: $\deg(b,c,u,w,\epsilon)=(1,0,1,0,1)$)

```
ρw = (0 1; 0 0); ρu = (0 0; -ϵ 0); ρb = (-1 0; 0 -1); ρc = (-(1+1/ϵ)/2 0; 0 (1-1/ϵ)/2); ρ0 = (0 0; 0 0);
B[x_?MatrixQ, y_?MatrixQ] := x.y - y.x;
Simplify@{B[ρw, ρc] == ρw, B[ρc, ρu] == ρu, B[ρu, ρw] == ρb - 2 ϵ ρc}
```

{True, True, True}

```
ρb - ϵ ρc // Simplify // MatrixForm
```

$$\begin{pmatrix} \frac12(-1+\epsilon) & 0 \\ 0 & \frac12(-1-\epsilon)\end{pmatrix}$$

```
MatrixForm /@ Simplify /@ ((ρb - ϵ ρc) ⊗ ρc)
```

$$\begin{pmatrix} \frac12(-1+\epsilon) & 0 \\ 0 & \frac12(-1-\epsilon)\end{pmatrix} \otimes \begin{pmatrix} -\frac{1+\epsilon}{2\epsilon} & 0 \\ 0 & \frac{-1+\epsilon}{2\epsilon}\end{pmatrix}$$

```
MatrixForm /@ Simplify /@ ((-2 ρb + 2 ϵ ρc) ⊗ (-2 ϵ ρc)) / (4 ϵ)
```

$$\frac{\begin{pmatrix} 1-\epsilon & 0 \\ 0 & 1+\epsilon\end{pmatrix} \otimes \begin{pmatrix} 1+\epsilon & 0 \\ 0 & 1-\epsilon\end{pmatrix}}{4\epsilon}$$

```
MatrixExp[a ρw].MatrixExp[b ρu] // Simplify // MatrixForm
```

$$\begin{pmatrix} 1-ab\epsilon & a \\ -b\epsilon & 1\end{pmatrix}$$

```
eqn = MatrixExp[a ρw].MatrixExp[b ρu] == MatrixExp[α ρu].MatrixExp[β ρw].MatrixExp[γ (ρb - 2 ϵ ρc)]
```

$\{\{1-ab\epsilon, a\}, \{-b\epsilon, 1\}\} == \{\{e^{\gamma\epsilon}, e^{-\gamma\epsilon}\beta\}, \{-e^{\gamma\epsilon}\alpha\epsilon, e^{-\gamma\epsilon}(1-\alpha\beta\epsilon)\}\}$

```
{sol} = Solve[Thread[Flatten /@ eqn], {α, β, γ}]
```

$\left\{\left\{\alpha\to-\frac{b}{-1+ab\epsilon}, \beta\to a-a^2b\epsilon, \gamma\to \text{ConditionalExpression}\left[\frac{2 i\pi C[1]+\text{Log}[1-ab\epsilon]}{\epsilon}, C[1]\in \text{Integers}\right]\right\}\right\}$

```
sol = sol /. C[1] → 0
```

$\left\{\alpha\to-\frac{b}{-1+ab\epsilon}, \beta\to a-a^2b\epsilon, \gamma\to\frac{\text{Log}[1-ab\epsilon]}{\epsilon}\right\}$

```
Series[{α, β, γ} /. sol, {ϵ, 0, 1}]
```

$\left\{b+ab^2\epsilon+O[\epsilon]^2, a-a^2b\epsilon+O[\epsilon]^2, -ab-\frac12(a^2b^2)\epsilon+O[\epsilon]^2\right\}$

```
Simplify[eqn /. sol]
```

True

```
Column@Series[{α, β, γ} /. sol, {ϵ, 0, 4}]
```

$b+ab^2\epsilon+a^2b^3\epsilon^2+a^3b^4\epsilon^3+a^4b^5\epsilon^4+O[\epsilon]^5$

$a-a^2b\epsilon+O[\epsilon]^5$

$-ab-\frac12(a^2b^2)\epsilon-\frac13(a^3b^3)\epsilon^2-\frac14(a^4b^4)\epsilon^3-\frac15(a^5b^5)\epsilon^4+O[\epsilon]^5$