

The Turbo-Gassner Representation

Pensieve header: The turbo Gassner representation. Continues MinimalGassnerPlus.nb and pensieve://Projects/OneCo-1604/.

The Gassner Representation

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Gi,j := ReplaceAll[#, vj => (1 - ti) vi + ti vj] &

R3l = {v1, v2, v3} // G1,2 // G1,3 // G2,3
{v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 ((1 - t2) v2 + t2 v3) }

R3r = {v1, v2, v3} // G2,3 // G1,3 // G1,2
{v1, (1 - t1) v1 + t1 v2, (1 - t2) ((1 - t1) v1 + t1 v2) + t2 ((1 - t1) v1 + t1 v3) }

R3l - R3r // Expand
{0, 0, 0}

OCl = {v1, v2, v3} // G1,2 // G1,3
{v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 v3}

OCr = {v1, v2, v3} // G1,3 // G1,2
{v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 v3}

OCl - OCr // Expand
{0, 0, 0}

UCl = {v1, v2, v3} // G1,3 // G2,3
{v1, v2, (1 - t1) v1 + t1 ((1 - t2) v2 + t2 v3) }

UCr = {v1, v2, v3} // G2,3 // G1,3
{v1, v2, (1 - t2) v2 + t2 ((1 - t1) v1 + t1 v3) }

UCl - UCr // Expand
{0, 0, v1 - t1 v1 - t2 v1 + t1 t2 v1 - v2 + t1 v2 + t2 v2 - t1 t2 v2}

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The Gassner-Plus Representation

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GPi,j := Composition[Expand,
  ReplaceAll[#, {wj => (1 - ti) wi + ti wj,
    f- . vj => f (1 - ti) vi + f ti vj + (ti - 1) (ti ∂ti f - tj ∂tj f) wi + f ti wi }]] &
]

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R31 =

$$\{f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] v_2, f[t_1, t_2, t_3] v_3, w_1, w_2, w_3\} // GP_{1,2} // GP_{1,3} // GP_{2,3}$$

$$\left\{ f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] t_1 v_1 + \right.$$

$$f[t_1, t_2, t_3] t_1 v_2 + f[t_1, t_2, t_3] t_1 w_1 + t_2 w_1 f^{(0,1,0)}[t_1, t_2, t_3] -$$

$$t_1 t_2 w_1 f^{(0,1,0)}[t_1, t_2, t_3] - t_1 w_1 f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2 w_1 f^{(1,0,0)}[t_1, t_2, t_3],$$

$$f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] t_1 v_1 + f[t_1, t_2, t_3] t_1 v_2 - f[t_1, t_2, t_3] t_1 t_2 v_2 +$$

$$f[t_1, t_2, t_3] t_1 t_2 v_3 + f[t_1, t_2, t_3] t_1 w_1 + f[t_1, t_2, t_3] t_1 t_2 w_2 +$$

$$t_3 w_1 f^{(0,0,1)}[t_1, t_2, t_3] - t_1 t_3 w_1 f^{(0,0,1)}[t_1, t_2, t_3] + t_1 t_3 w_2 f^{(0,0,1)}[t_1, t_2, t_3] -$$

$$t_1 t_2 t_3 w_2 f^{(0,0,1)}[t_1, t_2, t_3] - t_1 t_2 w_2 f^{(0,1,0)}[t_1, t_2, t_3] +$$

$$t_1 t_2^2 w_2 f^{(0,1,0)}[t_1, t_2, t_3] - t_1 w_1 f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2 w_1 f^{(1,0,0)}[t_1, t_2, t_3],$$

$$w_1, w_1 - t_1 w_1 + t_1 w_2, w_1 - t_1 w_1 + t_1 w_2 - t_1 t_2 w_2 + t_1 t_2 w_3 \}$$

R3r =

$$\{f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] v_2, f[t_1, t_2, t_3] v_3, w_1, w_2, w_3\} // GP_{2,3} // GP_{1,3} // GP_{1,2}$$

$$\left\{ f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] t_1 v_1 + \right.$$

$$f[t_1, t_2, t_3] t_1 v_2 + f[t_1, t_2, t_3] t_1 w_1 + t_2 w_1 f^{(0,1,0)}[t_1, t_2, t_3] -$$

$$t_1 t_2 w_1 f^{(0,1,0)}[t_1, t_2, t_3] - t_1 w_1 f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2 w_1 f^{(1,0,0)}[t_1, t_2, t_3],$$

$$f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] t_1 v_1 + f[t_1, t_2, t_3] t_1 v_2 - f[t_1, t_2, t_3] t_1 t_2 v_2 +$$

$$f[t_1, t_2, t_3] t_1 t_2 v_3 + f[t_1, t_2, t_3] t_1 w_1 + f[t_1, t_2, t_3] t_1 t_2 w_2 +$$

$$t_3 w_1 f^{(0,0,1)}[t_1, t_2, t_3] - t_1 t_3 w_1 f^{(0,0,1)}[t_1, t_2, t_3] + t_1 t_3 w_2 f^{(0,0,1)}[t_1, t_2, t_3] -$$

$$t_1 t_2 t_3 w_2 f^{(0,0,1)}[t_1, t_2, t_3] - t_1 t_2 w_2 f^{(0,1,0)}[t_1, t_2, t_3] +$$

$$t_1 t_2^2 w_2 f^{(0,1,0)}[t_1, t_2, t_3] - t_1 w_1 f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2 w_1 f^{(1,0,0)}[t_1, t_2, t_3],$$

$$w_1, w_1 - t_1 w_1 + t_1 w_2, w_1 - t_1 w_1 + t_1 w_2 - t_1 t_2 w_2 + t_1 t_2 w_3 \}$$

R31 - R3r

$$\{0, 0, 0, 0, 0, 0\}$$

OC1 = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, w1, w2, w3} // GP1,2 // GP1,3

$$\left\{ f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] t_1 v_1 + \right.$$

$$f[t_1, t_2, t_3] t_1 v_2 + f[t_1, t_2, t_3] t_1 w_1 + t_2 w_1 f^{(0,1,0)}[t_1, t_2, t_3] -$$

$$t_1 t_2 w_1 f^{(0,1,0)}[t_1, t_2, t_3] - t_1 w_1 f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2 w_1 f^{(1,0,0)}[t_1, t_2, t_3],$$

$$f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] t_1 v_1 + f[t_1, t_2, t_3] t_1 v_3 + f[t_1, t_2, t_3] t_1 w_1 +$$

$$t_3 w_1 f^{(0,0,1)}[t_1, t_2, t_3] - t_1 t_3 w_1 f^{(0,0,1)}[t_1, t_2, t_3] - t_1 w_1 f^{(1,0,0)}[t_1, t_2, t_3] +$$

$$t_1^2 w_1 f^{(1,0,0)}[t_1, t_2, t_3], w_1, w_1 - t_1 w_1 + t_1 w_2, w_1 - t_1 w_1 + t_1 w_3 \}$$

OCr = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, w1, w2, w3} // GP1,3 // GP1,2

$$\left\{ f[t_1, t_2, t_3] v_1, f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] t_1 v_1 + \right.$$

$$f[t_1, t_2, t_3] t_1 v_2 + f[t_1, t_2, t_3] t_1 w_1 + t_2 w_1 f^{(0,1,0)}[t_1, t_2, t_3] -$$

$$t_1 t_2 w_1 f^{(0,1,0)}[t_1, t_2, t_3] - t_1 w_1 f^{(1,0,0)}[t_1, t_2, t_3] + t_1^2 w_1 f^{(1,0,0)}[t_1, t_2, t_3],$$

$$f[t_1, t_2, t_3] v_1 - f[t_1, t_2, t_3] t_1 v_1 + f[t_1, t_2, t_3] t_1 v_3 + f[t_1, t_2, t_3] t_1 w_1 +$$

$$t_3 w_1 f^{(0,0,1)}[t_1, t_2, t_3] - t_1 t_3 w_1 f^{(0,0,1)}[t_1, t_2, t_3] - t_1 w_1 f^{(1,0,0)}[t_1, t_2, t_3] +$$

$$t_1^2 w_1 f^{(1,0,0)}[t_1, t_2, t_3], w_1, w_1 - t_1 w_1 + t_1 w_2, w_1 - t_1 w_1 + t_1 w_3 \}$$

OC1 - OCr

 $\{0, 0, 0, 0, 0, 0\}$

Question. Does GP factor through G? How?

The Turbo-Gassner Representation

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TGRule = {
  f_. v_j := f v_j + f u_{k,j,k} + ((1 - t_j^-1) (t_j ∂_{t_j} f - t_k ∂_{t_k} f) - f) u_{j,j,k},
  f_. v_k := f (1 - t_j) v_j + f t_j v_k -
    t_j^-1 (t_j - 1)^2 (t_j ∂_{t_j} f - t_k ∂_{t_k} f) u_{j,j,k} + (t_j - 1) (t_j ∂_{t_j} f - t_k ∂_{t_k} f) u_{k,j,k},
  f_. v_i := f v_i + (1 - t_j^-1) (t_j ∂_{t_j} f - t_k ∂_{t_k} f) u_{j,i,k},
  u_{j,j,j} := u_{j,j,j} + (1 - t_j^-1) u_{j,j,k},
  u_{j,j,k} := t_j^-1 u_{j,j,k},
  u_{j,k,j} | u_{k,j,j} := (1 - t_j) u_{j,j,j} - t_j^-1 (t_j - 1)^2 u_{j,j,k} + t_j u_{k,j,j} + (t_j - 1) u_{k,j,k},
  u_{j,k,k} | u_{k,j,k} := (t_j^-1 - 1) u_{j,j,k} + u_{k,j,k},
  u_{k,k,j} := (t_j - 1)^2 u_{j,j,j} + t_j^-1 (t_j - 1)^3 u_{j,j,k} -
    2 (t_j - 1) t_j u_{k,j,j} - 2 (t_j - 1)^2 u_{k,j,k} + t_j^2 u_{k,k,j} + (t_j - 1) t_j u_{k,k,k},
  u_{k,k,k} := t_j^-1 (t_j - 1)^2 u_{j,j,k} + (2 - 2 t_j) u_{k,j,k} + t_j u_{k,k,k},
  u_{j,j,l_} := u_{j,j,l},
  u_{j,k,l_} | u_{k,j,l_} := (1 - t_j) u_{j,j,l} + t_j u_{k,j,l},
  u_{k,k,l_} := (t_j - 1)^2 u_{j,j,l} - 2 (t_j - 1) t_j u_{k,j,l} + t_j^2 u_{k,k,l},
  u_{i_,j,j} | u_{j,i_,j} := u_{j,i,j} + (1 - t_j^-1) u_{j,i,k},
  u_{i_,j,k} | u_{j,i_,k} := t_j^-1 u_{j,i,k},
  u_{i_,k,j} | u_{k,i_,j} := (1 - t_j) u_{j,i,j} - t_j^-1 (t_j - 1)^2 u_{j,i,k} + t_j u_{k,i,j} + (t_j - 1) u_{k,i,k},
  u_{i_,k,k} | u_{k,i_,k} := (t_j^-1 - 1) u_{j,i,k} + u_{k,i,k},
  u_{j,i_,l_} | u_{i_,j,l_} := u_{j,i,l},
  u_{k,i_,l_} | u_{i_,k,l_} := (1 - t_j) u_{j,i,l} + t_j u_{k,i,l},
  u_{i1_,i2_,j} := u_{i2,i1,j} + (1 - t_j^-1) u_{i2,i1,k},
  u_{i1_,i2_,k} := t_j^-1 u_{i2,i1,k},
  u_{i1_,i2_,l_} := u_{i2,i1,l}];
Unset[TGj_,k_];
SetDelayed@{TGj_,k_, Expand@*
  ReplaceAll[u_{jj_,ii_,kk_} /; ! OrderedQ[{jj_, ii_}] := u_{ii_,jj_,kk_}]@*ReplaceAll[TGRule];
Unset::norep: Assignment on Subscript for TGj_,k_ not found. >>

checks = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1,1,1, u1,1,2, u1,2,1, u1,2,2,
  u1,2,1, u1,2,2, u2,2,1, u2,2,2, u1,1,3, u1,2,3, u1,2,3, u2,2,3, u0,1,1, u0,1,2, u0,2,1, u0,2,2,
  u0,1,1, u0,1,2, u0,2,1, u0,2,2, u0,1,3, u0,2,3, u0,1,3, u0,2,3, u0,1,1, u0,1,2, u0,1,3, u3,3,3};
Short[R31 = checks // TG1,2 // TG1,3 // TG2,3]
{<<30>> + t1 u1,1,3 f^{(1,0,0)}[t1, t2, t3], <<1>>, <<1>>,
  <<25>>, <<1>>,  $\frac{u_{<<1>>}}{t_1 <<1>>}$ ,  $-\frac{2 u_{1,1,3}}{t_2} + <<18>> + t_1 t_2 u_{3,3,3}$ }

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The Turbo-Burau Representation

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η /: η[i_]2 = 0;
η /: η[i_] η[j_] = 0;
ε0[f_] := f / . _η → 0;
εj,k[f_] := Coefficient[f, η[j]] - Coefficient[f, η[k]];
TBRule = {
  f_. vj ⇒ f vj + ε0[f] uk,j,k + ((t - 1) εj,k[f] - ε0[f]) uj,j,k,
  f_. vk ⇒
    f (1 - t - η[j]) vj + f (t + η[j]) vk - (t - 1)2 εj,k[f] uj,j,k + t (t - 1) εj,k[f] uk,j,k,
  f_. vi ⇒ f vi + (t - 1) εj,k[f] uj,i,k,
  uj,j,j ⇒ uj,j,j + (1 - t-1) uj,j,k,
  uj,j,k ⇒ t-1 uj,j,k,
  uj,k,j | uk,j,j ⇒ (1 - t) uj,j,j - t-1 (t - 1)2 uj,j,k + t uk,j,j + (t - 1) uk,j,k,
  uj,k,k | uk,j,k ⇒ (t-1 - 1) uj,j,k + uk,j,k,
  uk,k,j ⇒ (t - 1)2 uj,j,j + t-1 (t - 1)3 uj,j,k -
    2 (t - 1) t uk,j,j - 2 (t - 1)2 uk,j,k + t2 uk,k,j + (t - 1) t uk,k,k,
  uk,k,k ⇒ t-1 (t - 1)2 uj,j,k + (2 - 2 t) uk,j,k + t uk,k,k,
  uj,j,l ⇒ uj,j,l,
  uj,k,l | uk,j,l ⇒ (1 - t) uj,j,l + t uk,j,l,
  uk,k,l ⇒ (t - 1)2 uj,j,l - 2 (t - 1) t uk,j,l + t2 uk,k,l,
  ui,j,j | uj,i,j ⇒ uj,i,j + (1 - t-1) uj,i,k,
  ui,j,k | uj,i,k ⇒ t-1 uj,i,k,
  ui,k,j | uk,i,j ⇒ (1 - t) uj,i,j - t-1 (t - 1)2 uj,i,k + t uk,i,j + (t - 1) uk,i,k,
  ui,k,k | uk,i,k ⇒ (t-1 - 1) uj,i,k + uk,i,k,
  uj,i,l | ui,j,l ⇒ uj,i,l,
  uk,i,l | ui,k,l ⇒ (1 - t) uj,i,l + t uk,i,l,
  ui1,i2,j ⇒ ui2,i1,j + (1 - t-1) ui2,i1,k,
  ui1,i2,k ⇒ t-1 ui2,i1,k,
  ui1,i2,l ⇒ ui2,i1,l};
SetDelayed@@{TBj,k, Expand@*
  ReplaceAll[ujj,ii,kk /; ! OrderedQ[{jj, ii}] ⇒ uii,jj,kk]@*ReplaceAll[TBRule]};

ff = f0 + f1 η[1] + f2 η[2] + f3 η[3];
checks = {ff v1, ff v2, ff v3, u1,1,1, u1,1,2, u1,2,1, u1,2,2, u1,2,1, u1,2,2,
  u2,2,1, u2,2,2, u1,1,3, u1,2,3, u1,2,3, u2,2,3, u0,1,1, u0,1,2, u0,2,1, u0,2,2, u0,1,1,
  u0,1,2, u0,2,1, u0,2,2, u0,1,3, u0,2,3, u0,1,3, u0,2,3, u0,1,1, u0,1,2, u0,1,3};
Short[R31 = checks // TB1,2 // TB1,3 // TB2,3]

{f0 v1 - f0 u1,1,2 - f1 u1,1,2 + t f1 u1,1,2 + <<24>> + f1 v1 η[1] + f2 v1 η[2] + f3 v1 η[3],
  <<1>>, <<26>>, <<1>>,  $\frac{u_{<<1>>}}{t^2}$ }

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