

Abstract. The commutator of two elements x and y in a group G is $xyx^{-1}y^{-1}$. That is, x followed by y followed by the inverse of x followed by the inverse of y . In my talk I will tell you how commutators are related to the following four riddles:

1. Can you send a secure message to a person you have never communicated with before (neither privately nor publicly), using a messenger you do not trust?
2. Can you hang a picture on a string on the wall using n nails, so that if you remove any one of them, the picture will fall?
3. Can you draw an n -component link (a knot made of n non-intersecting circles) so that if you remove any one of those n components, the remaining $(n - 1)$ will fall apart?
4. Can you solve the quintic in radicals? Is there a formula for the zeros of a degree 5 polynomial in terms of its coefficients, using only the operations on a scientific calculator?

Definition. The commutator of two elements x and y in a group G is $[x, y] := xyx^{-1}y^{-1}$.

Example 1. In S_3 , $[(12), (23)] = (12)(23)(12)^{-1}(23)^{-1} = (123)$ and in general in $S_{\geq 3}$,

$$[(ij), (jk)] = (ijk).$$

Example 2. In $S_{\geq 4}$,

$$[(ijk), (jkl)] = (ijk)(jkl)(ijk)^{-1}(jkl)^{-1} = (il)(jk).$$

Example 3. In $S_{\geq 5}$,

$$[(ijk), (klm)] = (ijk)(klm)(ijk)^{-1}(klm)^{-1} = (jkm).$$

The Princess Bride, 1987.

Inigo: You are using Bonetti's defense against me, uh?

Man in Black: I thought it fitting, considering the rocky terrain.

Inigo: Naturally, you must expect me to attack with Capo Ferro.

MiB: Naturally, but I find that Thibault cancels out Capo Ferro, don't you?

Inigo: Unless the enemy has studied his Agrippa, which I have! You are wonderful!

MiB: Thank you. I've worked hard to become so.

Inigo: I admit it, you are better than I am.

MiB: Then why are you smiling?

Inigo: Because I know something you don't know.

MiB: And what is that? **Inigo:** I am not left-handed.

MiB: You're amazing! **Inigo:** I ought to be after twenty years.

MiB: There is something I ought to tell you. **Inigo:** Tell me.

MiB: I'm not left-handed either. **Inigo:** Who are you?

MiB: No one of consequence. **Inigo:** I must know.

MiB: Get used to disappointment. **Inigo:** Okay.

Inigo: Kill me quickly.

MiB: I would as soon destroy a stained-glass window as an artist like yourself. However, since I can't have you following me either. . .

MiB: Please understand I hold you in the highest respect.

Solving the Quadratic, $ax^2 + bx + c = 0$: $\delta = \sqrt{\Delta}$; $\Delta = b^2 - 4ac$;
 $r = \frac{\delta - b}{2a}$.

Solving the Cubic, $ax^3 + bx^2 + cx + d = 0$: $\Delta = 27a^2d^2 - 18abcd + 4ac^3 + 4b^3d - b^2c^2$; $\delta = \sqrt{\Delta}$; $\Gamma = 27a^2d - 9abc + 3\sqrt{3}a\delta + 2b^3$;

$$\gamma = \sqrt[3]{\frac{\Gamma}{2}}; r = -\frac{b^2 - 3ac + b + \gamma}{3a}.$$

Solving the Quartic, $ax^4 + bx^3 + cx^2 + dx + e = 0$: $\Delta_0 = 12ae - 3bd + c^2$; $\Delta_1 = -72ace + 27ad^2 + 27b^2e - 9bcd + 2c^3$;
 $\Delta_2 = \frac{1}{27}(\Delta_1^2 - 4\Delta_0^3)$; $u = \frac{8ac - 3b^2}{8a^2}$; $v = \frac{8a^2d - 4abc + b^3}{8a^3}$; $\delta_2 = \sqrt{\Delta_2}$;

$$Q = \frac{1}{2}(3\sqrt{3}\delta_2 + \Delta_1); q = \sqrt[3]{Q}; S = \frac{\Delta_0 + q}{12a} - \frac{u}{6}; s = \sqrt{S};$$

$$\Gamma = -\frac{v}{s} - 4S - 2u; \gamma = \sqrt{\Gamma}; r = -\frac{b}{4a} + \frac{\gamma}{2} + s.$$

Yes, Prime Minister, 1986.

Sir Humphrey: You know what happens: nice young lady comes up to you. Obviously you want to create a good impression, you don't want to look a fool, do you? So she starts asking you some questions: Mr. Woolley, are you worried about the number of young people without jobs?

Bernard Woolley: Yes

H: Are you worried about the rise in crime among teenagers?

W: Yes

H: Do you think there is a lack of discipline in our Comprehensive schools?

W: Yes

H: Do you think young people welcome some authority and leadership in their lives?

W: Yes

H: Do you think they respond to a challenge?

W: Yes

H: Would you be in favour of reintroducing National Service?

W: Oh...well, I suppose I might be.

H: Yes or no? **W:** Yes

H: Of course you would, Bernard. After all you told me can't say no to that. So they don't mention the first five questions and they publish the last one.

W: Is that really what they do?

H: Well, not the reputable ones no, but there aren't many of those. So alternatively the young lady can get the opposite result.

W: How?

H: Mr. Woolley, are you worried about the danger of war?

W: Yes

H: Are you worried about the growth of armaments? **W:** Yes

H: Do you think there is a danger in giving young people guns and teaching them how to kill? **W:** Yes

H: Do you think it is wrong to force people to take up arms against their will? **W:** Yes

H: Would you oppose the reintroduction of National Service?

W: Yes

H: There you are, you see Bernard. The perfect balanced sample.

References.

(Arnold, 1960s, hard to locate)

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