Pensieve Header: An ad hoc unitarity property for Γ-calculus.

```
dir = SetDirectory["C:/drorbn/AcademicPensieve/2014-06/"];
<< KnotTheory`
<< "MetaCalculi/MetaCalculi-Program.m"
ΓSimp = Factor;
```

```
Loading KnotTheory` version of April 3, 2014, 16:23:56.0784.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
t1 = Xp[1, 2] ** Xm[3, 1] ** Xp[2, 3] ** Xm[1, 2] ** Xp[3, 1] ** Xm[2, 3] // Γ
```

$$\begin{pmatrix} 1 & s_1 & s_2 \\ s_1 & -\frac{-1+T_2-T_1 T_2+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3}{T_1} & \frac{(-1+T_1)\,(1-T_2+T_1 T_2)\,(-1+T_3\cdots}{T_1} \\ s_2 & \frac{(-1+T_2)\,(-1+T_3)\,(-1+T_3-T_1 T_3-T_2 T_3+T_1 T_2 T_3)}{T_1 T_2 T_3} & -\frac{1-T_1-T_2+T_1 T_2-2 T_3+2 T_1 T_3+3 T_2 T_3-5 T_1 T_2 T_3+T_1^2 T_2 T_3-T_2^2 T_3+2 T_1 T_2^2 T_3-T_1^2 T_2^2 T_3+T_3^2\cdots}{T_1 T_2 T_3} \\ s_3 & \frac{(-1+T_2)\,(-1+T_3)\,(1-T_3+T_1 T_3)}{T_1 T_2 T_3} & -\frac{(-1+T_1)\,(-1+T_3)\,(-1+T_2+T_3-T_2 T_3\cdots}{T_1 T_2 T_3} \\ \Sigma & 1 & 1 \end{pmatrix}$$

```
MatrixForm[A1 = Simplify[t1@A /. T_ → T]]
```

$$\begin{pmatrix} -2+\frac{1}{T}+3T-T^2 & -3+\frac{1}{T}+4T-3T^2+T^3 & -(-1+T)^2 \\ \frac{(-1+T)^2\,(-1+T-2T^2+T^3)}{T^3} & 9-\frac{1}{T^3}+\frac{4}{T^2}-\frac{7}{T}-7T+4T^2-T^3 & \frac{(-1+T)^2\,(1-T+T^2)}{T^2} \\ \frac{(-1+T)^2\,(1-T+T^2)}{T^3} & -\frac{(-1+T)^2\,(-1+2T-T^2+T^3)}{T^3} & -2-\frac{1}{T^2}+\frac{3}{T}+T \end{pmatrix}$$

```
Ω[n_] := Table[Which[i < j, 0, i == j, 1, i > j, 1 - T], {i, n}, {j, n}];
Ω[3] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 1-T & 1 & 0 \\ 1-T & 1-T & 1 \end{pmatrix}$$

```
Simplify[Transpose[A1].Ω[3].(A1 /. T → 1 / T)] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 1-T & 1 & 0 \\ 1-T & 1-T & 1 \end{pmatrix}$$

```
Ωc[n_] := Table[Which[i < j, 0, i == j, 1/(1 - T_i), i > j, 1], {i, n}, {j, n}];
Ωc[3] // MatrixForm
```

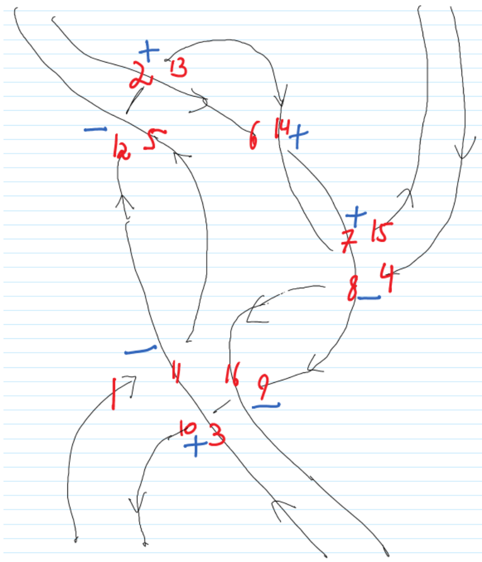
$$\begin{pmatrix} \frac{1}{1-T_1} & 0 & 0 \\ 1 & \frac{1}{1-T_2} & 0 \\ 1 & 1 & \frac{1}{1-T_3} \end{pmatrix}$$

```
MatrixForm[A1c = Simplify[t1@A]]
```

$$\begin{pmatrix} \frac{1-(-1+T_1)\,T_2\,(-1+T_3)+(-1+T_1)\,T_3}{T_1} & \frac{(-1+T_1)\,(1+(-1+T_1)\,T_2)\,(-1+T_3)}{T_1} \\ \frac{(-1+T_2)\,(-1+T_3)\,(-1+(-1+T_1)\,(-1+T_2)\,T_3)}{T_1 T_2 T_3} & \frac{-T_1^2\,(-1+T_2)\,T_2\,(-1+T_3)\,T_3-(-1+T_2)\,(-1+T_3)\,(1+(-1+T_2)\,T_3)+T_1\left((-1+T_3)^2+2T_2^2\,(-1+T_3)\,T_3+T\cdots\right.}{T_1 T_2 T_3} \\ \frac{(-1+T_2)\,(-1+T_3)\,(1+(-1+T_1)\,T_3)}{T_1 T_2 T_3} & -\frac{(-1+T_1)\,(-1+T_3)\,(-1+T_3+T_2\,(1+(-1+T_1)\,T_3))}{T_1 T_2 T_3} \end{pmatrix}$$

```
Simplify[Transpose[A1c].Ωc[3].(A1c /. T_i_ → 1 / T_i)] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{1-T_1} & 0 & 0 \\ 1 & \frac{1}{1-T_2} & 0 \\ 1 & 1 & \frac{1}{1-T_3} \end{pmatrix}$$




```
γ0 = Xm[11, 1] Xm[5, 12] Xp[2, 13] Xp[14, 6] Xp[7, 15] Xm[8, 4] Xm[16, 9] Xp[3, 10] // Γ;
γ1 =
  γ0 // dm[1, 5, 1] // dm[2, 6, 2] // dm[2, 7, 2] // dm[2, 8, 2] // dm[2, 9, 2] // dm[2, 10,
        2] // dm[3, 11, 3] // dm[3, 12, 3] //
     dm[3, 13, 3] // dm[3, 14, 3] // dm[3, 15, 3] // dm[4, 16, 4];
γ2 = γ1 // dS[2] // dS[4] ;
MatrixForm[A2 = Simplify[γ2@A /. T_ → T]]
```

$$\begin{pmatrix} 1+\frac{1}{T^2}-\frac{1}{T} & \frac{(-1+T)^2}{T^2} & \frac{-1+T}{T} & 0 \\ \frac{2\,(-1+T)^2}{T^2\,(-1+2\,T)} & \frac{3-5\,T+3\,T^2}{T^2\,(-1+2\,T)} & \frac{2-2\,T}{T-2\,T^2} & \frac{-1+T}{-1+2\,T} \\ \frac{-1+T}{T^2\,(-1+2\,T)} & \frac{(-1+T)\,(1+T^2)}{T^3\,(-1+2\,T)} & \frac{1}{T\,(-1+2\,T)} & \frac{(-1+T)^2}{T\,(-1+2\,T)} \\ 0 & \frac{-1+T}{T^3} & 0 & \frac{1}{T} \end{pmatrix}$$

```
MatrixForm /@ {Ω[4], Simplify[Transpose[A2].Ω[4].(A2 /. T → 1 / T)]}
```

$$\left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-T & 1 & 0 & 0 \\ 1-T & 1-T & 1 & 0 \\ 1-T & 1-T & 1-T & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-T & 1 & 0 & 0 \\ 1-T & 1-T & 1 & 0 \\ 1-T & 1-T & 1-T & 1 \end{pmatrix}\right\}$$

```
Ω[4] // Inverse // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1+T & 1 & 0 & 0 \\ -T+T^2 & -1+T & 1 & 0 \\ -T^2+T^3 & -T+T^2 & -1+T & 1 \end{pmatrix}$$

```
MatrixForm[A2c = Simplify[γ2@A]]
```

$$\begin{pmatrix} 1+\frac{-1+\frac{1}{T_3}}{T_1} & \frac{(-1+T_1)(-1+T_3)}{T_1 T_3} & 1-\frac{1}{T_1} & 0 \\ \frac{(-1+T_2)(-1+T_3)(T_2+T_3)}{T_1 T_2 T_3 (-1+T_2+T_3)} & \frac{(-1+T_2)(-1+T_3)(T_2+T_3)T_4+T_1 T_2 (1+(-1+T_2)T_4)}{T_1 T_2 T_3 (-1+T_2+T_3) T_4} & \frac{(-1+T_2)(T_2+T_3)}{T_1 T_2 (-1+T_2+T_3)} & \frac{-1+T_2}{-1+T_2+T_3} \\ \frac{-1+T_3}{T_1 T_2 (-1+T_2+T_3)} & \frac{(-1+T_3)(T_3 T_4+T_1(1+(-1+T_2)T_4))}{T_1 T_2 T_3 (-1+T_2+T_3) T_4} & \frac{T_3}{T_1 T_2 (-1+T_2+T_3)} & \frac{(-1+T_2)(-1+T_3)}{T_2(-1+T_2+T_3)} \\ 0 & \frac{-1+T_4}{T_2 T_3 T_4} & 0 & \frac{1}{T_2} \end{pmatrix}$$

```
MatrixForm /@ {Ωc[4], Simplify[Transpose[A2c].Ωc[4].(A2c /. T_i_ → 1 / T_i)]}
```

$$\left\{\begin{pmatrix} \frac{1}{1-T_1} & 0 & 0 & 0 \\ 1 & \frac{1}{1-T_2} & 0 & 0 \\ 1 & 1 & \frac{1}{1-T_3} & 0 \\ 1 & 1 & 1 & \frac{1}{1-T_4} \end{pmatrix}, \begin{pmatrix} \frac{1}{1-T_1} & 0 & 0 & 0 \\ 1 & \frac{1}{1-T_2} & 0 & 0 \\ 1 & 1 & \frac{1}{1-T_3} & 0 \\ 1 & 1 & 1 & \frac{1}{1-T_4} \end{pmatrix}\right\}$$

```
γ0 = Xm[11, 1] Xm[5, 12] Xp[2, 13] Xp[14, 6] Xp[7, 15] Xm[8, 4] Xm[16, 9] Xp[3, 10] // Γ;
γ1 =
  γ0 // dm[1, 5, 1] // dm[2, 6, 2] // dm[2, 7, 2] // dm[2, 8, 2] // dm[2, 9, 2] // dm[2, 10,
        2] // dm[3, 11, 3] // dm[3, 12, 3] //
     dm[3, 13, 3] // dm[3, 14, 3] // dm[3, 15, 3] // dm[4, 16, 4];
γ2 = γ1 // dS[2] // dS[4] ;
MatrixForm[A2 = Simplify[γ2@A]]
```

$$\begin{pmatrix} 1+\frac{-1+\frac{1}{T_3}}{T_1} & \frac{(-1+T_1)(-1+T_3)}{T_1 T_3} & 1-\frac{1}{T_1} & 0 \\ \frac{(-1+T_2)(-1+T_3)(T_2+T_3)}{T_1 T_2 T_3 (-1+T_2+T_3)} & \frac{(-1+T_2)(-1+T_3)(T_2+T_3)T_4+T_1 T_2 (1+(-1+T_2)T_4)}{T_1 T_2 T_3 (-1+T_2+T_3) T_4} & \frac{(-1+T_2)(T_2+T_3)}{T_1 T_2 (-1+T_2+T_3)} & \frac{-1+T_2}{-1+T_2+T_3} \\ \frac{-1+T_3}{T_1 T_2 (-1+T_2+T_3)} & \frac{(-1+T_3)(T_3 T_4+T_1(1+(-1+T_2)T_4))}{T_1 T_2 T_3 (-1+T_2+T_3) T_4} & \frac{T_3}{T_1 T_2 (-1+T_2+T_3)} & \frac{(-1+T_2)(-1+T_3)}{T_2(-1+T_2+T_3)} \\ 0 & \frac{-1+T_4}{T_2 T_3 T_4} & 0 & \frac{1}{T_2} \end{pmatrix}$$

```
Ωi[n_] := Table[Which[i < j, 0, i == j, 1, i > j, 1 - T_i], {i, n}, {j, n}];
Ωj[n_] := Table[Which[i < j, 0, i == j, 1, i > j, 1 - T_j], {i, n}, {j, n}];
MatrixForm /@ {Ωi[4], Ωj[4]}
```

$$\left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-T_2 & 1 & 0 & 0 \\ 1-T_3 & 1-T_3 & 1 & 0 \\ 1-T_4 & 1-T_4 & 1-T_4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-T_1 & 1 & 0 & 0 \\ 1-T_1 & 1-T_2 & 1 & 0 \\ 1-T_1 & 1-T_2 & 1-T_3 & 1 \end{pmatrix}\right\}$$

```
MatrixForm /@ {Ωi[4], FullSimplify[Transpose[A2].Ωi[4].(A2 /. T_i_ → 1 / T_i)]}
```

$$\left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-T_2 & 1 & 0 & 0 \\ 1-T_3 & 1-T_3 & 1 & 0 \\ 1-T_4 & 1-T_4 & 1-T_4 & 1 \end{pmatrix},\right.$$

$T_1^2\, T_2\, (T_2\, (-1+\ldots$

$-\dfrac{(T_2 (-1+T_3)-T_3)(-1+T_3)\left((-1+T_2)T_2^2+(-1+T_2)T_2 T_3+T_3^2\right)T_4+T_1^2 T_2(-1+T_3)\left(T_2(T_2-T_3)T_3+\left(T_2^3 T_3\ldots\right.\right.}{\ldots}$

$T_1^2\ldots$

```
MatrixForm /@ {Ωj[4], FullSimplify[Transpose[A2].Ωj[4].(A2 /. T_i_ → 1 / T_i)]}
```

$$\left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1-T_1 & 1 & 0 & 0 \\ 1-T_1 & 1-T_2 & 1 & 0 \\ 1-T_1 & 1-T_2 & 1-T_3 & 1 \end{pmatrix},\right.$$

$\dfrac{-T_2^3(-1+T_3)^2-(-1+T_3)T_3^2+T_1(T_2(-1+T_3)-T_3)(-1+T_3)(-1+T_2+T_3)-T_2^2(-1+T_3)}{(T_2(-1+T_3)-T_3)T_3(-1+T_2+T_3)}$

$\dfrac{T_1 T_2(-1+T_3)\left(-T_2^2+T_3^2\right)-(-1+T_1)\left(-T_2^3(-1+T_3)^2-(-1+T_3)T_3^2+T_1(T_2(-1+T_3)-T_3)(-1+T_3)(-1+T_2+\ldots\right.}{(T_2(-1+T_3)-T_3)T_3(-1+T_2+T_3)T_4}$

$-\dfrac{(-1+T_3)\left(T_3+T_2\left((-1+T_2)^2+(-3+T_2)T_3\ldots\right.\right.}{(T_2(-1+T_3)-T_3)(-1+T_2+T_3)}$

$\dfrac{T_1 T_2^3(-1+T_3)^2+\left(-1+T_1+T_1^2(-1+T_3)\right)(-1+T_3)T_3+T_2^2(-1+T_3)\left(1+T_1\left(T_1-(2+T_1)T_3+T_3^2\right)\right)+T_2\left(1+(-3+\ldots\right.}{(T_2(-1+T_3)-T_3)(-1+T_2+T_3)}$

```
Xp[a, b] // A
```

$$\begin{pmatrix} 1 & h[b] \\ t[a] & \frac{-1+e^{c_a}}{c_a} \end{pmatrix}$$

```
{n = 3; Ωc[3] // MatrixForm,
 γ0 = Γ[ω, Σ_{a=0}^{n} h_a σ_a, Σ_{a=1}^{n} Σ_{b=1}^{n} t_a h_b Ωc[n][[a, b]]],
 γ0 // A}
```

$$\left\{\begin{pmatrix} \frac{1}{1-T_1} & 0 & 0 \\ 1 & \frac{1}{1-T_2} & 0 \\ 1 & 1 & \frac{1}{1-T_3} \end{pmatrix}, \begin{pmatrix} \omega & s_1 & s_2 & s_3 \\ s_1 & -\frac{1}{-1+T_1} & 0 & 0 \\ s_2 & 1 & -\frac{1}{-1+T_2} & 0 \\ s_3 & 1 & 1 & -\frac{1}{-1+T_3} \\ \Sigma & \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \frac{1-\sigma_1+e^{c_1}\sigma_1}{-c_1+e^{c_1}c_1} & 0 & 0 \\ t[2] & -\frac{1}{c_2} & \frac{1-\sigma_2+e^{c_2}\sigma_2}{-c_2+e^{c_2}c_2} & 0 \\ t[3] & -\frac{1}{c_3} & -\frac{1}{c_3} & \frac{1-\sigma_3+e^{c_3}\sigma_3}{-c_3+e^{c_3}c_3} \end{pmatrix}\right\}$$

$$\frac{1-\sigma_1+e^{c_1}\sigma_1}{-c_1+e^{c_1}c_1}\,c_1-1\ \text{// Simplify}$$

$$\frac{2-e^{c_1}+\left(-1+e^{c_1}\right)\sigma_1}{-1+e^{c_1}}$$

$$\left(\frac{2 - e^{c_1} + (-1 + e^{c_1})\,\sigma_1}{-1 + e^{c_1}} \;/.\; \sigma_1 \to e^{c_1}\right) \;//\; \texttt{Simplify}$$

$$\frac{2 - 2\,e^{c_1} + e^{2\,c_1}}{-1 + e^{c_1}}$$

$$\texttt{Solve}\left[\frac{2 - e^{c_1} + (-1 + e^{c_1})\,\sigma_1}{-1 + e^{c_1}} == \sigma_1,\ \sigma_1\right]$$

$$\{\}$$

$$\Big\{n = 3;\ \left(\Omega c[3] \prod_{a=1}^{n} (1 - T_a)\right) \;//\; \texttt{MatrixForm},$$

$$\gamma 1 = \Gamma\left[\omega,\ \sum_{a=0}^{n} h_a T_a,\ \left(\prod_{a=1}^{n} (1 - T_a)\right) \left(\sum_{a=1}^{n}\sum_{b=1}^{n} t_a\, h_b\, \Omega c[n][[a, b]]\right)\right],$$

$$\gamma 1 \;//\; A \;//\; \alpha\texttt{Collect}[\texttt{Factor}]\Big\}$$

$$\left\{\begin{pmatrix} (1 - T_2)\,(1 - T_3) & 0 & 0 \\ (1 - T_1)\,(1 - T_2)\,(1 - T_3) & (1 - T_1)\,(1 - T_3) & 0 \\ (1 - T_1)\,(1 - T_2)\,(1 - T_3) & (1 - T_1)\,(1 - T_2)\,(1 - T_3) & (1 - T_1)\,(1 - T_2) \end{pmatrix},\right.$$

$$\begin{pmatrix} \omega & s_1 & s_2 & s_3 \\ s_1 & (-1 + T_2)\,(-1 + T_3) & 0 & 0 \\ s_2 & -(-1 + T_1)\,(-1 + T_2)\,(-1 + T_3) & (-1 + T_1)\,(-1 + T_3) & 0 \\ s_3 & -(-1 + T_1)\,(-1 + T_2)\,(-1 + T_3) & -(-1 + T_1)\,(-1 + T_2)\,(-1 + T_3) & (-1 + T_1)\,(-1 + T_2) \\ \Sigma & T_1 & T_2 & T_3 \end{pmatrix},$$

$$\left.\begin{pmatrix} \omega & h[1] & h[2] & h[3] \\ t[1] & \frac{-1 + e^{c_1} + e^{c_2} + e^{c_3} - e^{c_2 + c_3}}{c_1} & 0 & 0 \\ t[2] & \frac{-1 + e^{c_1} + e^{c_2} - e^{c_1 + c_2} + e^{c_3} - e^{c_1 + c_3} - e^{c_2 + c_3} + e^{c_1 + c_2 + c_3}}{c_2} & \frac{-1 + e^{c_1} + e^{c_2} + e^{c_3} - e^{c_1 + c_3}}{c_2} & 0 \\ t[3] & \frac{-1 + e^{c_1} + e^{c_2} - e^{c_1 + c_2} + e^{c_3} - e^{c_1 + c_3} - e^{c_2 + c_3} + e^{c_1 + c_2 + c_3}}{c_3} & \frac{-1 + e^{c_1} + e^{c_2} - e^{c_1 + c_2} + e^{c_3} - e^{c_1 + c_3} - e^{c_2 + c_3} + e^{c_1 + c_2 + c_3}}{c_3} & \frac{-1 + e^{c_1} + e^{c_2} - e^{c_1 + c_2} + e^{c_3}}{c_3} \end{pmatrix}\right\}$$

$$\texttt{Series}\left[\frac{-1 + e^{c_1} + e^{c_2} - e^{c_1 + c_2} + e^{c_3} - e^{c_1 + c_3} - e^{c_2 + c_3} + e^{c_1 + c_2 + c_3}}{c_3},\ \{c_3, 0, 2\}\right]$$

$$(-1 + e^{c_1})\,(-1 + e^{c_2}) + \frac{1}{2}\,(-1 + e^{c_1})\,(-1 + e^{c_2})\,c_3 + \frac{1}{6}\,(-1 + e^{c_1})\,(-1 + e^{c_2})\,c_3^2 + O[c_3]^3$$