A NOTE ON THE UNITARITY PROPERTY OF THE GASSNER INVARIANT

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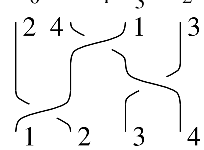
Abstract. We give a 3-page description of the Gassner invariant (or representation) of braids (or pure braids), along with a description and a proof of its unitarity property.

The unitarity of the Gassner representation [Ga] of the pure braid group was discussed by many authors (e.g. [Lo, Ab, KLW]) and from several points of view, yet without exposing how utterly simple the formulas turn out to be[1]. When the present author needed quick and easy formulas, he couldn't find them. This note is written in order to rectify this situation (but with no discussion of theory). I was heavily influenced by a similar discussion of the unitarity of the Burau representation in [KT, Section 3.1.2].

Let n be a natural number. The braid group B_n on n strands is the group with generators σ_i, for $1 \le i \le n-1$, and with relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ when $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ when $1 \le i \le n-2$. A standard way to depict braids, namely elements of B_n, appears on the right. Braids are made of strands that are indexed 1 through n at the bottom. The generator σ_i denotes a positive crossing between the strand at position #i as counted just below the horizontal level of that crossing, and the strand just to its right. Note that with the strands indexed at the bottom, the two strands participating in a crossing corresponding to σ_i may have arbitrary indices, depending on the permutation induced by the braids below the level of that crossing.

$b_0 = \sigma_1\sigma_3^{-1}\sigma_2$:



Let t be a formal variable and let $U_i(t) = U_{n;i}(t)$ denote the $n\times n$ identity matrix with its 2×2 block at rows i and $i+1$ and columns i and $i+1$ replaced by $\begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix}$. Let $U_i^{-1}(t)$ be the inverse of $U_i(t)$; it is the $n\times n$ identity matrix with the block at $\{i, i+1\}\times\{i, i+1\}$ replaced by $\begin{pmatrix} 0 & \bar{t} \\ 1 & 1-\bar{t} \end{pmatrix}$, where $\bar{t}$ denotes t^{-1}.

$$U_{5;3}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1-t & 1 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let b be a braid $b = \prod_{\alpha=1}^k \sigma_{i_\alpha}^{s_\alpha}$, where the s_α are signs and where products are taken from left to right. Let j_α be the index of the "over" strand at crossing #α in b. The Gassner invariant $\Gamma(b)$ of b is given by the formula on the right. It is a Laurent polynomial in n formal variables $t_1, \ldots, t_n$, with coefficients in $\mathbb{Z}$.

$$\Gamma(b) := \prod_{\alpha=1}^{k} U_{i_\alpha}^{s_\alpha}(t_{j_\alpha})$$

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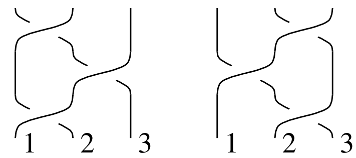
[1]Partially this is because the formulas are simplest when extended a "Gassner invariant" defined on the full braid group, but then it is not a representation and it is not unitary. Yet it has an easy "unitarity property"; see below.

For example, $\Gamma(\sigma_1\sigma_2\sigma_1) = U_1(t_1)U_2(t_1)U_1(t_2)$ while $\Gamma(\sigma_2\sigma_1\sigma_2) = U_2(t_2)U_1(t_1)U_2(t_1)$. The equality of these two matrix products constitutes the bulk of the proof of the well-definedness of Γ, and the rest is even easier. The verification of this equality is a routine exercise in 3×3 matrix multiplication. Impatient readers may find it in the *Mathematica* notebook that accompanies this note, [BN].

$\sigma_1\sigma_2\sigma_1$: $\sigma_2\sigma_1\sigma_2$:



A second example is the braid b_0 of the first figure. Here and in [BN],

$$\Gamma(b_0) = U_1(t_1)U_3^{-1}(t_4)U_2(t_1) = \begin{pmatrix} 1-t_1 & 1-t_1 & 1 & 0 \\ t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{t}_4 \\ 0 & t_1 & 0 & 1-\bar{t}_4 \end{pmatrix}$$

Given a permutation $\tau = [\tau 1, \ldots, \tau n]$ of $1, \ldots, n$, let $\Omega(\tau)$ be the triangular $n \times n$ matrix shown on the right (diagonal entries $(1-t_{\tau i})^{-1}$, 1's below the diagonal, 0's above). Let ι denote the identity permutation $[1, 2, \ldots, n]$.

$$\Omega(\tau) := \begin{pmatrix} (1-t_{\tau 1})^{-1} & 0 & \cdots & 0 \\ 1 & (1-t_{\tau 2})^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & (1-t_{\tau n})^{-1} \end{pmatrix}$$

Theorem. Let b be a braid that induces a strand permutation $\tau = [\tau 1, \ldots, \tau n]$ (meaning, the strand indices that appear at the top of b are $\tau 1, \tau 2, \ldots, \tau n$). Let $\gamma = \Gamma(b)$ be the Gassner invariant of b. Then γ satisfies the "unitarity property"

$$\Omega(\tau)\gamma^{-1} = \bar{\gamma}^T\Omega(\iota), \qquad \text{or equivalently,} \qquad \gamma^{-1} = \Omega(\tau)^{-1}\bar{\gamma}^T\Omega(\iota), \tag{1}$$

where $\bar{\gamma}$ is γ subject to the substitution $\forall i\; t_i \to \bar{t}_i := t_i^{-1}$, and $\bar{\gamma}^T$ is the transpose matrix of $\bar{\gamma}$.

Proof. A direct and simple-minded computation proves Equation (1) for $b = \sigma_i$ and for $b = \sigma_i^{-1}$, namely for $\gamma = U_i(t_i)$ and for $\gamma = U_i^{-1}(t_{i+1})$ (impatient readers see [BN]), and then, clearly, using the second form of Equation (1), the statement generalizes to products with all the intermediate $\Omega(\tau)^{-1}\Omega(\tau)$ pairs cancelling out nicely. □

If the Gassner invariant Γ is restricted to pure braids, namely to braids that induce the identity permutation, it becomes multiplicative and then it can be called "the Gassner representation" (in general Γ can be recast as a homomorphism into $M_{n\times n}(\mathbb{Z}[t_i, \bar{t}_i]) \rtimes S_n$, where S_n acts on matrices by permuting the variables t_i appearing in their entries).

For pure braids $\Omega(\tau) = \Omega(\iota) =: \Omega$ and hence by conjugating (in the $t_i \to 1/t_i$ sense) and transposing Equation (1) and replacing γ by γ^{-1}, we find that the theorem also holds if Ω is replaced by $\bar{\Omega}^T$. Hence, extending the coefficients to $\mathbb{C}$, the theorem also holds if Ω is replaced by $\Psi := i\Omega - i\bar{\Omega}^T$, which is formally Hermitian ($\bar{\Psi}^T = \Psi$).

If the t_i's are specialized to complex numbers of unit norm then inversion is the same as complex conjugation. If also the t_i's are sufficiently close to 1 and have positive imaginary parts, then Ψ is dominated by its main diagonal entries, which are real, positive, and large, and hence Ψ is positive definite and genuinely Hermitian. Thus in that case, the Gassner representation is unitary in the standard sense of the word, relative to the inner product on $\mathbb{C}^n$ defined by Ψ.

We remark is that the Gassner representation easily extends to a representation of pure v/w-braids. See e.g. [BND, Sections 2.1.2 and 2.2], where the generators σ_{ij} are described (they are *not* generators of the ordinary pure braid group). Simply set $\Gamma(\sigma_{ij})^{\pm 1} = U_{ij}^{\pm 1}$ where U_{ij} is the $n \times n$ identity matrix with its 2×2 block at rows i and j and columns i and j replaced by $\begin{pmatrix} 1 & 1-t_i \\ 0 & t_i \end{pmatrix}$. Yet on v/w-braids Γ does not satisfy the unitarity property of this note and I'd be very surprised if it is at all unitary.

We also remark that there is an alternative form Γ' for the Gassner representation of pure v/w-braids, defined by $\Gamma'(\sigma_{ij})^{\pm 1} = V_{ij}^{\pm 1}$ where V_{ij} is the $n \times n$ identity matrix with its 2×2 block at rows i and j and columns i and j replaced by $\begin{pmatrix} 1 & 1-t_j \\ 0 & t_i \end{pmatrix}$. Clearly, U_{ij} and V_{ij} are conjugate; $V_{ij} = D^{-1}U_{ij}D$, with D the diagonal matrix whose (i,i) entry is $1 - t_i$ for every i. Hence on ordinary pure braids and for appropriate values of the t_i's (as above), Γ' is also unitary, relative to the Hermitian inner product defined by the matrix

$$\Psi' := \bar{D}^T \Psi D = i\bar{D}^T(\Omega - \bar{\Omega}^T)D$$

whose printed form is better avoided (yet it appears at the end of [BN]).

References

[Ab] M. N. Abdulrahim, *A Faithfulness Criterion for the Gassner Representation of the Pure Braid Group,* Proceedings of the American Mathematical Society **125-5** (1997) 1249–1257.

[BN] D. Bar-Natan, `UnitarityOfGassnerDemo.nb`, a *Mathematica* noteboook at `http://drorbn.net/AcademicPensieve/2014-06/UnitarityOfGassner/`.

[BND] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of w-Knotted Objects I: w-Knots and the Alexander Polynomial,* `http://drorbn.net/AcademicPensieve/Projects/WKO1/` and arXiv:1405.1956.

[Ga] B. J. Gassner, *On Braid Groups,* Ph.D. thesis, New York Univeristy, 1959.

[KT] C. Kassel and V. Turaev, *Braid Groups,* Springer GTM **247**, 2008.

[KLW] P. Kirk, C. Livingston, and Z. Wang, *The Gassner Representation for String Links,* Communications in Contemporary Mathematics **3-1** (2001) 87–136, arXiv:math/9806035.

[Lo] D. D. Long, *On the Linear Representation of Braid Groups,* Transactions of the American Mathematical Society **311-2** (1989) 535–560.

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Pensieve header: Mathemaica notebook accompanying “A Note on the Unitarity Property of the Gassner Invariant” by Dror Bar-Natan, http://drorbn.net/AcademicPensieve/2014-06/UnitarityOf-Gassner/.

Definitions.

```
U_i_[t_] := ReplacePart[IdentityMatrix[n],
   {{i, i} → 1 - t, {i, i + 1} → 1,
    {i + 1, i} → t, {i + 1, i + 1} → 0}];
U-_i_[t_] := Inverse[U_i[t]];
Ω[τ___] := Table[
   Which[i < j, 0, i == j, (1 - t_{τ}[[i]])^-1, i > j, 1],
   {i, n}, {j, n}];
OverBar[X_] := X /. t_i_ :→ 1 / t_i;
U_i_,j_ := ReplacePart[IdentityMatrix[n],
   {{i, i} → 1, {i, j} → 1 - t_i,
    {j, i} → 0, {j, j} → t_i}];
V_i_,j_ := ReplacePart[IdentityMatrix[n],
   {{i, i} → 1, {i, j} → 1 - t_j,
    {j, i} → 0, {j, j} → t_i}];
DD := DiagonalMatrix[Table[1 - t_i, {i, n}]];
```

The named matrices.

```
n = 5; MatrixForm /@ Simplify /@ {U_3[t], U-_3[t]}
```

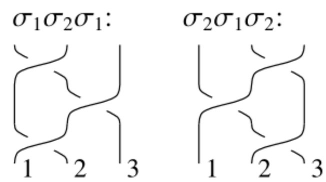
$$\left\{\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1-t&1&0\\0&0&t&0&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&\frac{1}{t}&0\\0&0&1&\frac{-1+t}{t}&0\\0&0&0&0&1\end{pmatrix}\right\}$$

```
n = 3; MatrixForm /@ Simplify /@ {Ω[2, 3, 1], Inverse[Ω[2, 3, 1]]}
```

$$\left\{\begin{pmatrix}\frac{1}{1-t_2}&0&0\\1&\frac{1}{1-t_3}&0\\1&1&\frac{1}{1-t_1}\end{pmatrix},\begin{pmatrix}1-t_2&0&0\\-(-1+t_2)\,(-1+t_3)&1-t_3&0\\-(-1+t_1)\,(-1+t_2)\,t_3&-(-1+t_1)\,(-1+t_3)&1-t_1\end{pmatrix}\right\}$$

```
n = 5; MatrixForm /@ {U_4,1, V_4,1, DD}
```

$$\left\{\begin{pmatrix}t_4&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\1-t_4&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}t_4&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\1-t_1&0&0&1&0\\0&0&0&0&1\end{pmatrix},\begin{pmatrix}1-t_1&0&0&0&0\\0&1-t_2&0&0&0\\0&0&1-t_3&0&0\\0&0&0&1-t_4&0\\0&0&0&0&1-t_5\end{pmatrix}\right\}$$

The R3 move  .


```
n = 3; MatrixForm /@ Simplify /@ {U1[t1].U2[t1].U1[t2], U2[t2].U1[t1].U2[t1]}
```

$$\left\{\begin{pmatrix} 1-t_1 & 1-t_1 & 1 \\ -t_1\,(-1+t_2) & t_1 & 0 \\ t_1\,t_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1-t_1 & 1-t_1 & 1 \\ -t_1\,(-1+t_2) & t_1 & 0 \\ t_1\,t_2 & 0 & 0 \end{pmatrix}\right\}$$

The unitarity property for the generators.

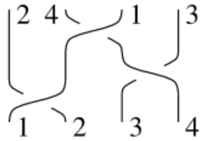
```
n = 5; γ = U3[t3];
MatrixForm /@
 Simplify /@ {Ω[1, 2, 4, 3, 5].Inverse[γ], Transpose[γ̄].Ω[1, 2, 3, 4, 5]}
```

$$\left\{\begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{t_3-t_3\,t_4} & 0 \\ 1 & 1 & \frac{1}{1-t_3} & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}, \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{1}{t_3-t_3\,t_4} & 0 \\ 1 & 1 & \frac{1}{1-t_3} & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}\right\}$$

```
n = 5; γ = Ū3[t4];
MatrixForm /@
 FullSimplify /@ {Ω[1, 2, 4, 3, 5].Inverse[γ], Transpose[γ̄].Ω[1, 2, 3, 4, 5]}
```

$$\left\{\begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{1}{1-t_4} & 0 \\ 1 & 1 & 1-\frac{t_3\,t_4}{-1+t_3} & 1 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}, \begin{pmatrix} \frac{1}{1-t_1} & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{1-t_2} & 0 & 0 & 0 \\ 1 & 1 & 1 & \frac{1}{1-t_4} & 0 \\ 1 & 1 & 1-\frac{t_3\,t_4}{-1+t_3} & 1 & 0 \\ 1 & 1 & 1 & 1 & \frac{1}{1-t_5} \end{pmatrix}\right\}$$

The braid $b_0 = \sigma_1\sigma_3^{-1}\sigma_2$:



.

```
n = 4; MatrixForm[γ0 = U1[t1].U3bar[t4].U2[t1]]
```

$$\begin{pmatrix} 1-t_1 & 1-t_1 & 1 & 0 \\ t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t_4} \\ 0 & t_1 & 0 & -\frac{1-t_4}{t_4} \end{pmatrix}$$

The unitarity property for b_0.

```
MatrixForm /@ Simplify /@ {Ω[2, 4, 1, 3].Inverse[γ0], Transpose[γ0bar].Ω[1, 2, 3, 4]}
```

$$\left\{\begin{pmatrix} 0 & \frac{1}{t_1-t_1 t_2} & 0 & 0 \\ 0 & \frac{1}{t_1} & \frac{1}{t_1} & \frac{1}{t_1-t_1 t_4} \\ \frac{1}{1-t_1} & 0 & 0 & 0 \\ 1 & 1 & -\frac{1+t_3(-1+t_4)}{-1+t_3} & 1 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{t_1-t_1 t_2} & 0 & 0 \\ 0 & \frac{1}{t_1} & \frac{1}{t_1} & \frac{1}{t_1-t_1 t_4} \\ \frac{1}{1-t_1} & 0 & 0 & 0 \\ 1 & 1 & -\frac{1+t_3(-1+t_4)}{-1+t_3} & 1 \end{pmatrix}\right\}$$

On to w-braids

```
n = 3; MatrixForm /@ Simplify /@ {U1,2.U1,3.U2,3, U2,3.U1,3.U1,2}
```

$$\left\{\begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & -t_1(-1+t_2) \\ 0 & 0 & t_1 t_2 \end{pmatrix}, \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & -t_1(-1+t_2) \\ 0 & 0 & t_1 t_2 \end{pmatrix}\right\}$$

```
n = 3; MatrixForm /@ Simplify /@ {U1,2.U1,3, U1,3.U1,2}
```

$$\left\{\begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{pmatrix}, \begin{pmatrix} 1 & 1-t_1 & 1-t_1 \\ 0 & t_1 & 0 \\ 0 & 0 & t_1 \end{pmatrix}\right\}$$

The “Other Gassner” Γ’

```
n = 4; MatrixForm /@ Simplify /@ {V4,1, Inverse[DD].U4,1.DD}
```

$$\left\{\begin{pmatrix} t_4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-t_1 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} t_4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-t_1 & 0 & 0 & 1 \end{pmatrix}\right\}$$

```
n = 4; MatrixForm /@ Simplify /@ {
   Transpose[DD̄].Ω[1, 2, 3, 4].DD,
   Ψ' = ⅈ Transpose[DD̄].(Ω[1, 2, 3, 4] - Transpose[Ω[1, 2, 3, 4]̄]).DD
  }
```

$$\left\{\begin{pmatrix} 1-\frac{1}{t_1} & 0 & 0 & 0 \\ (1-t_1)\left(1-\frac{1}{t_2}\right) & 1-\frac{1}{t_2} & 0 & 0 \\ (1-t_1)\left(1-\frac{1}{t_3}\right) & (1-t_2)\left(1-\frac{1}{t_3}\right) & 1-\frac{1}{t_3} & 0 \\ (1-t_1)\left(1-\frac{1}{t_4}\right) & (1-t_2)\left(1-\frac{1}{t_4}\right) & (1-t_3)\left(1-\frac{1}{t_4}\right) & 1-\frac{1}{t_4} \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} \frac{i\left(-1+t_1^2\right)}{t_1} & i\left(-1+\frac{1}{t_1}\right)(1-t_2) & i\left(-1+\frac{1}{t_1}\right)(1-t_3) & i\left(-1+\frac{1}{t_1}\right)(1-t_4) \\ -\frac{i(-1+t_1)(-1+t_2)}{t_2} & \frac{i\left(-1+t_2^2\right)}{t_2} & i\left(-1+\frac{1}{t_2}\right)(1-t_3) & i\left(-1+\frac{1}{t_2}\right)(1-t_4) \\ -\frac{i(-1+t_1)(-1+t_3)}{t_3} & -\frac{i(-1+t_2)(-1+t_3)}{t_3} & \frac{i\left(-1+t_3^2\right)}{t_3} & i\left(-1+\frac{1}{t_3}\right)(1-t_4) \\ -\frac{i(-1+t_1)(-1+t_4)}{t_4} & -\frac{i(-1+t_2)(-1+t_4)}{t_4} & -\frac{i(-1+t_3)(-1+t_4)}{t_4} & \frac{i\left(-1+t_4^2\right)}{t_4} \end{pmatrix}\right\}$$