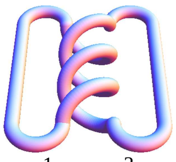
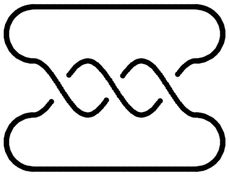
Knots in Four Dimensions and the Simplest Open Problem About Them

Abstract. I will describe a few 2-dimensional knots in 4 dimensional space in detail, then tell you how to make many more, then tell you that I don't really understand my way of making them, yet I can tell at least some of them apart in a colourful way.

u-Knots.

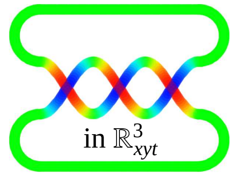


$S^1 \hookrightarrow \mathbb{R}^3_{xyz}$

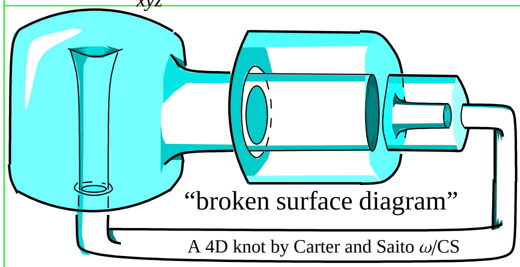


"broken arc diagram"

"thermographical diagram"

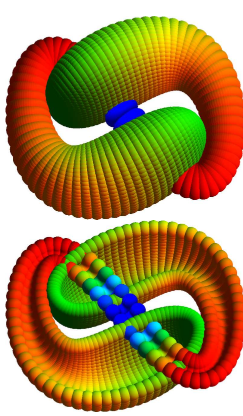


in $\mathbb{R}^3_{xyt}$



A 4D knot by Carter and Saito ω/CS

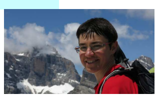
2-Knots.



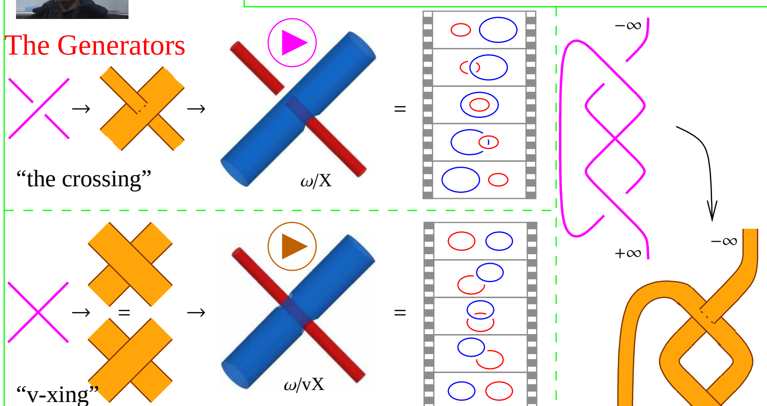
ω/F

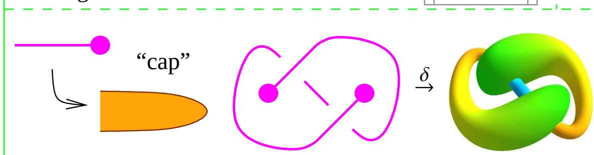
Satoh

Dalvit ω/Dal



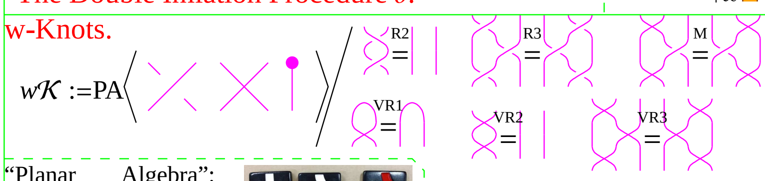
The Generators



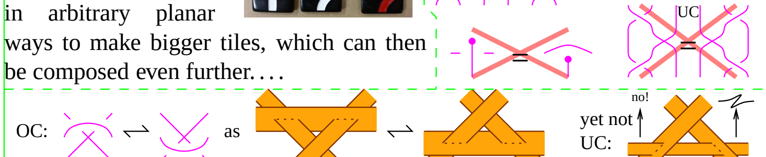


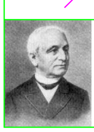
The Double Inflation Procedure δ.

w-Knots.



"Planar Algebra": The objects are "tiles" that can be composed in arbitrary planar ways to make bigger tiles, which can then be composed even further....

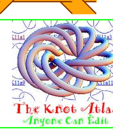




"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org



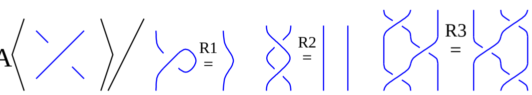
Satoh's Conjecture.

(ω/Sat) The "kernel" of the double inflation map δ, mapping w-knot diagrams in the plane to knotted 2D tubes and spheres in 4D, is precisely the moves R2-3, VR1-3, M, CP and OC listed above. In other words, two w-knot diagrams represent via δ the same 2D knot in 4D iff they differ by a sequence of the said moves.

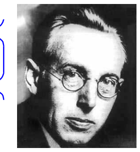
First Isomorphism Thm: $\delta\colon G \to H \;\Rightarrow\; \operatorname{im}\delta \cong G/\ker(\delta)$

δ is a map from algebra to topology. So a thing in "hard" topology ($\operatorname{im}\delta$) is the same as a thing in "easy" algebra ($w\mathcal{K}$).

Reidemeister's Theorem.



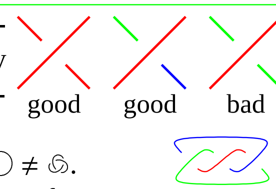
Proof by a genericity / "shaking" argument



Kurt Reidemeister

3-Colourings.

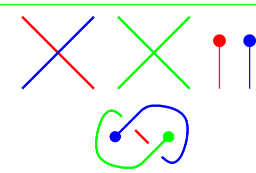
Colour the arcs of a broken arc diagram in RGB so that every crossing is either mono-chromatic or tri-chromatic; $\lambda(K) := |\{\text{3-colourings}\}|$.



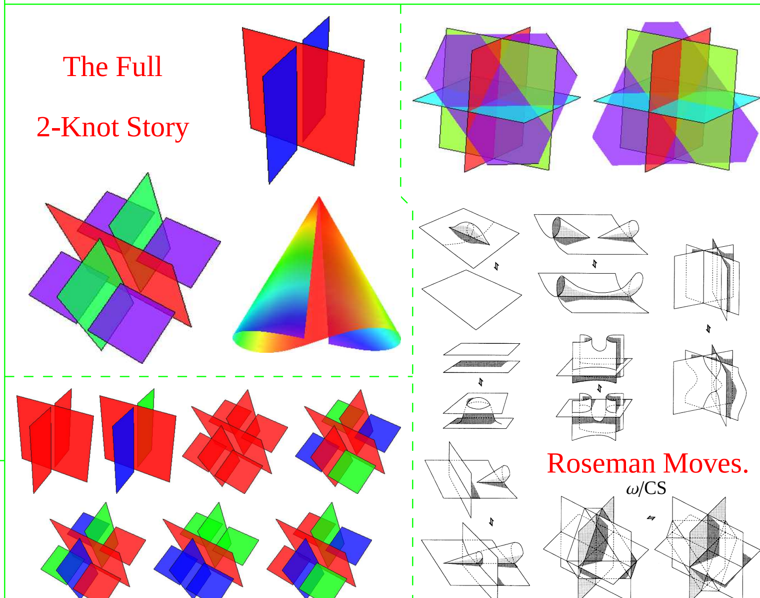
Example. $\lambda(\bigcirc) = 3$ while $\lambda(\text{trefoil}) = 9$; so $\bigcirc \neq$ trefoil.

Exercise. Show that the set of colourings of K is a vector space over $\mathbb{F}_3$ hence $\lambda(K)$ is always a power of 3.

Extend λ to $w\mathcal{K}$ by declaring that arcs "don't see" v-xings, and that caps are always "kosher". Then $\lambda(\bullet\!\!-\!\!\bullet) = 3 \neq 9 = \lambda(\text{CS 2-knot})$, so assuming Conjecture, the CS 2-knot is indeed knotted.



The Full 2-Knot Story



Roseman Moves.

ω/CS

Expansions.

Given a "ring" K and an ideal $I \subset K$, set

$$A := I^0/I^1 \oplus I^1/I^2 \oplus I^2/I^3 \oplus \cdots .$$

A homomorphic expansion is a multiplicative $Z\colon K \to A$ such that if $\gamma \in I^m$, then $Z(\gamma) = (0, 0, \ldots, 0, \gamma/I^{m+1}, *, *, \ldots)$.

Example. Let $K = C^\infty(\mathbb{R}^n)$ be smooth functions on $\mathbb{R}^n$, and $I := \{f \in K\colon f(0) = 0\}$. Then $I^m = \{f\colon f \text{ vanishes as } |x|^m\}$ and I^m/I^{m+1} is {homogeneous polynomials of degree m} and A is the set of power series. So Z is "a Taylor expansion".

Hence Taylor expansions are vastly general; even knots can be Taylor expanded!